Continua as minimal sets of homeomorphisms of S^2

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Abstract. Let f be an orientation preserving homeomorphism of S^2 which has a (nontrivial) continuum X as a minimal set. Then there are exactly two connected components of $S^2 \setminus X$ which are left invariant by f and all the others are wandering. The Carathéodory rotation number of an invariant component is irrational.

1. Introduction

An irrational rotation on S^2 has a round circle as a minimal set. Besides this a pathological diffeomorphism of S^2 is constructed in [**Ha**] which has a pseudo-circle as a minimal set. See also [**He**] for a curious diffeomorphism. These are the only examples known to the authors of homeomorphisms which have nontrivial continua as minimal sets. So a natural question to ask is if they are everything. The purpose of this paper is to present a modest partial answer to this question.

Let f be an orientation preserving homeomorphism of S^2 which has a (nontrivial) continuum X as a minimal set. A connected component U of $S^2 \setminus X$ is called an *invariant domain* if $fU = U$, a *periodic domain* if $f^nU = U$ for some $n \geq 1$, and a *wandering domain* otherwise.

Theorem 1.1. *There are exactly two invariant domains and all the other domains are wandering. The Carathéodory rotation number of an invariant domain is irrational.*

Sections 2 and 3 are expositions of the prime end theory and the Cartwright-Littlewood theorem, which are included since neither seems to have no short and self-contained expositions and some special features remarked in these sections are needed in the development of Section 4, which is devoted to the proof of Theorem 1.1. Both Sections 2 and 3 concern simply connected domains of closed oriented surfaces of any genus, and Section 4 orientation preserving homeomorphisms of the sphere S^2. In Section 5 we will construct a homeomorphism which actually admits a wandering domain.

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2. Prime ends

Denote by Σ a closed oriented surface equipped with a smooth Riemannian metric g and the area form *dvol*. Let $U \subset \Sigma$ be a *hyperbolic domain* i. e. an open simply connected subset such that $\Sigma \setminus U$ is not a singleton. (A nonhyperbolic simply connected domain exists only in the 2-sphere.) The purpose of this section is to show that a homeomorphism of U which extends to a homeomorphism of the closure $\overline{U}$ does extend to a homeomorphism of the so called Carathéodory compactification $\hat{U}$, a closed disc. Here we are only concerned with a simply connected domain in Σ. But there are generalization to more general domains, which are found in [**E**] and [**M**]. The proof of the main lemma (Lemma 2.2) is taken from [**E**].

Let $0 \in U$ be a base point. A real line properly embedded in U and not passing through 0 is called a *cross cut*. A cross cut c separates U into two hyperbolic domains, as can be seen by considering the one point compactification of U and applying the Jordan curve theorem. One of them not containing 0 is called the *content* of c and denoted by $U(c)$. A sequence of cross cuts $\{c_i\}_{i=1}^{\infty}$ is called a *chain* if $c_{i+1} \subset U(c_i)$ for each i. Two chains $\{c_i\}$ and $\{c'_i\}$ are called *equivalent* if for any i, there is j such that $c'_j \subset U(c_i)$ and $c_j \subset U(c'_i)$. An equivalence class of chains is called an *end* of U. A homeomorphism between two hyperbolic domains induces in an obvious way a bijection between the sets of ends. Given an end ξ, the relatively closed set $C(\xi) = \cap_i U(c_i)$ is independent of the choice of a chain $\{c_i\}$ from the end ξ, and is called the *content* of ξ.

A chain $\{c_i\}$ is called *topological* if the closures $\overline{c}_i$ of c_i in Σ are mutually disjoint and the diameter $\mathrm{diam}(c_i)$ converges to 0 as $i \to \infty$. Examples of topological chains are given in Figure 1. An end is called *prime* if it admits a topological chain.

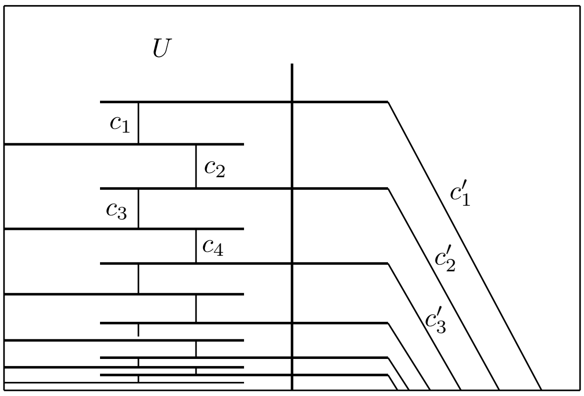

FIGURE 1. Topological chains

LEMMA 2.1. *The content $C(\xi)$ of a prime end ξ is empty.*

Proof. Assume the contrary and choose a point x from $C(\xi)$. Consider an arc γ in U joining 0 to x. See Figure 2. Then the distance from a point in γ to $\Sigma \setminus U$ is a continuous function on γ, and thus has a positive minimum. This contradicts the assumption that ξ is prime. $\square$

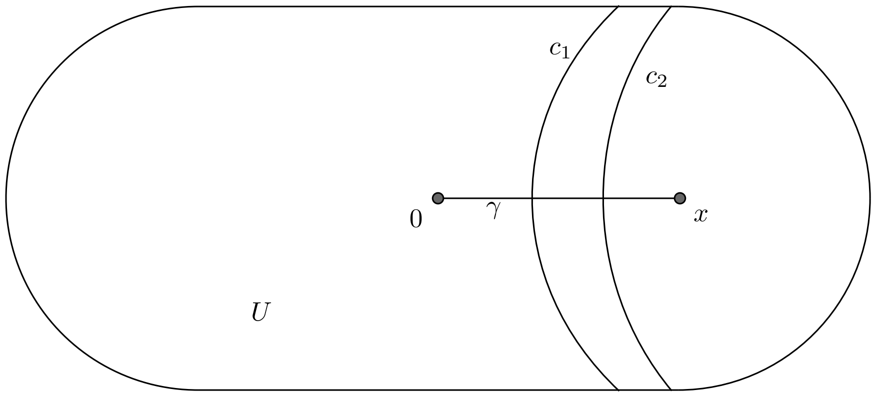

FIGURE 2

A positive valued continuous function ρ on U is called *admissible* if

$$\int_U \rho^2 dvol < \infty.$$

Given a subset c in U, ρ-diam(c) denotes the diameter of c w. r. t. the Riemannian metric $\rho^2 g$. (Function theorists often denotes the same metric by $\rho|dz|$.) An end ξ is called *conformal* if for any admissible function ρ there is a chain $\{c_i\}$ representing ξ such that ρ-diam$(c_i) \to 0$.

If $\phi : U \to V$ is a conformal equivalence and if $\rho : V \to (0, \infty)$ is admissible, then a function $\sigma : U \to (0, \infty)$ defined by $\sigma(z) = \rho(\phi(z))|\phi'(z)|$ is admissible, and for $c \subset V$, we have ρ-diam$(c) = \sigma$-diam$(\phi^{-1}(c))$. This shows that ϕ induces a bijection between the sets of the conformal ends of the two hyperbolic domains.

LEMMA 2.2. *An end ξ is prime if and only if it is conformal.*

Proof. First of all assuming that ξ is a prime end which is represented by a topological chain $\{c_i\}$, we shall show that ξ is a conformal end. By passing to a subsequence one may further assume that $\overline{c}_i$ converges to a point x_0. Since x_0 belongs to at most one $\overline{c}_i$, one may also assume that $x_0 \notin \overline{c}_i$ for any i. Take a polar coordinates (r, θ) around x_0. Let ρ be an arbitrary admissible function on U, extended to the whole Σ by letting $\rho = 0$ outside U. Then by the Schwarz inequality

$$\left(\int_0^\epsilon \int_0^{2\pi} \rho(r, \theta) r d\theta dr\right)^2 \leq \pi\epsilon^2 \cdot \int_{r \leq \epsilon} \rho^2 dvol.$$

Since ρ is admissible, $\int_{r \leq \epsilon} \rho^2 dvol \to 0$ as $\epsilon \to 0$, and we have

$$\frac{1}{\epsilon} \int_0^\epsilon \int_0^{2\pi} \rho(r, \theta) r d\theta dr \to 0 \quad (\epsilon \to 0).$$

Therefore we can find a sequence $\epsilon_k \downarrow 0$ such that

$$\int_0^{2\pi} \rho(\epsilon_k, \theta)\epsilon_k d\theta \to 0 \quad (k \to \infty).$$

Notice that the LHS above coincides with the ρ-length of the union of arcs $\{r = \epsilon_k\} \cap U$.

Now from the sequences $\{c_i\}$ and $\{\epsilon_k\}$, let us construct subsequences $\{c'_i\}$ and $\{\epsilon'_k\}$ by the following fashion. See Figure 3. First define $c'_1 = c_1$ and choose ϵ'_1

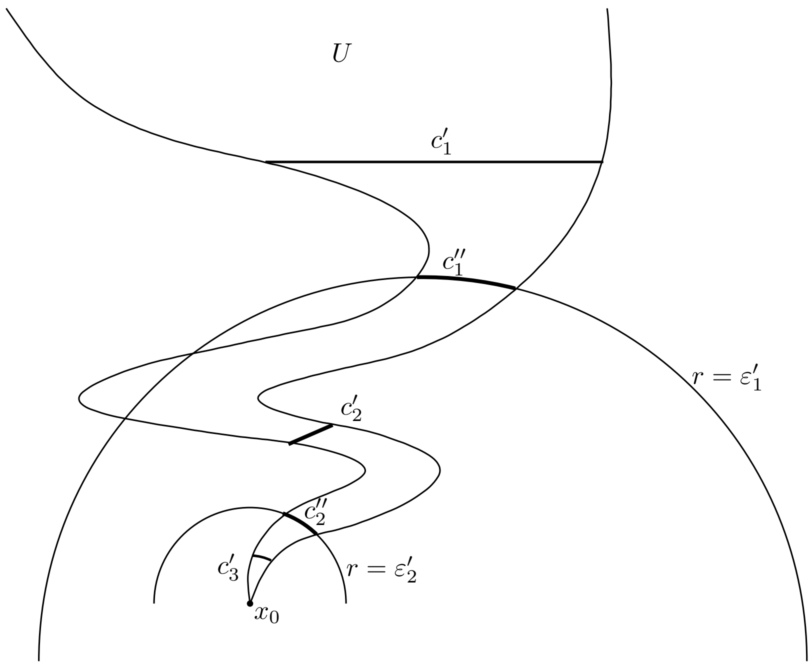

FIGURE 3

to be any ϵ_k from the sequence such that $\overline{c}'_1 \cap \{r \leq \epsilon'_1\} = \emptyset$. Then choose c'_2 to be any c_i from the sequence such that $\overline{c}'_2 \subset \{r < \epsilon'_1\}$. Next choose ϵ'_2 such that $\overline{c}'_2 \cap \{r \leq \epsilon'_2\} = \emptyset$, c'_3 such that $\overline{c}'_3 \subset \{r < \epsilon'_2\}$, and so forth.

Then there is a connected component c''_i of $\{r = \epsilon'_i\} \cap U$ which separates the cross cut c'_{i+1} from c'_i. To see this, construct a graph Γ; the vertices are connected components of $U \setminus \{r = \epsilon'_i\}$ and the edges connected components of $U \cap \{r = \epsilon'_i\}$. See figure 4. By transversality argument any two distinct vertices can be joined by a finite edge path. Actually Γ is a tree, since U is simply connected and any edge corresponds to a cross cut of U. Thus there is a unique shortest edge path joining the two vertices corresponding to the components, one containing c'_i, the other c'_{i+1}. The component c''_i of $U \cap \{r = \epsilon'_i\}$ corresponding to any edge of σ separates c'_{i+1} from c'_i. Clearly the chains $\{c'_i\}$ and $\{c''_i\}$ are equivalent and the latter satisfies ρ-diam$(c''_i) \to 0$, showing that ξ is conformal.

Next assume that ξ is conformal. First of all if we choose an admissible function ρ_0 which is constantly equal to 1 on U, we can find a chain $\{c_i\}$ such that $\mathrm{diam}(c_i) \to 0$ as $i \to \infty$. Passing to a subsequence if necessary, one may assume $c_i \to x_0$. Again let (r, θ) be the polar coordinates around x_0. Define a function ρ by

$$\rho(r, \theta) = -\frac{1}{r \log r}$$

if $r \leq 1/2$ and equal to $2/\log 2$ otherwise. Computation shows that the restriction of ρ to U is admissible. Now for any small $\epsilon > \delta$, the ρ-distance of the ϵ-circle and

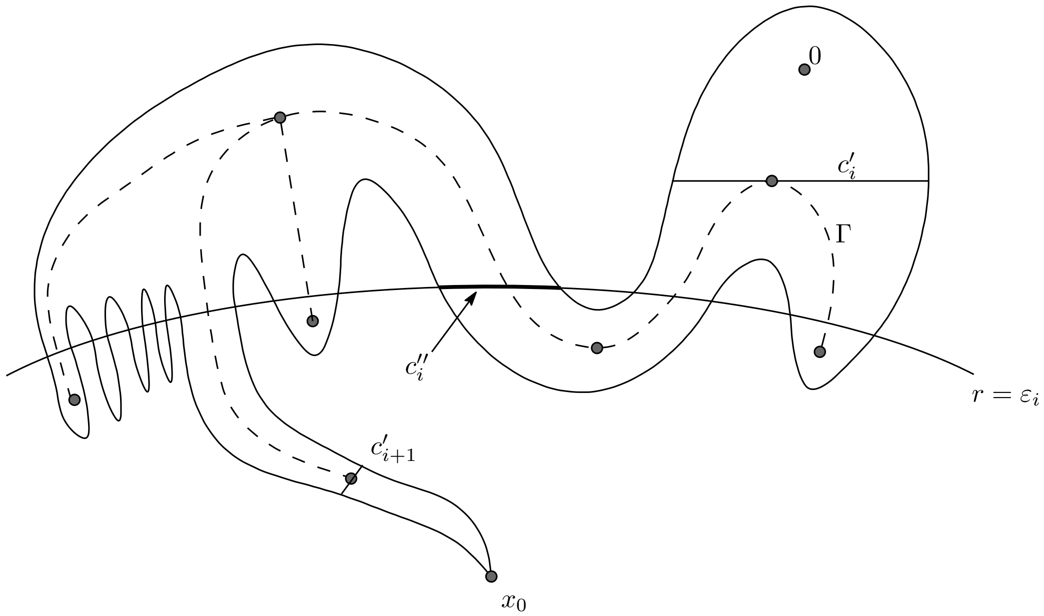

FIGURE 4

the δ-circle is given by

$$-\int_\delta^\epsilon \frac{dr}{r\log r} = \log(\log\delta/\log\epsilon),$$

which diverges to ∞ if we fix ϵ and let $\delta \to 0$. Let c_i' be a chain representing ξ such that ρ-diam$(c_i') \to 0$. Since ρ is bigger than a constant multiple of ρ_0, this implies also that diam$(c_i') \to 0$.

First consider the case where c_i' converges to x_0 (passing to a subsequence). See Figure 5. The above computation shows that for i big enough $\overline{c}_i'$ is a compact

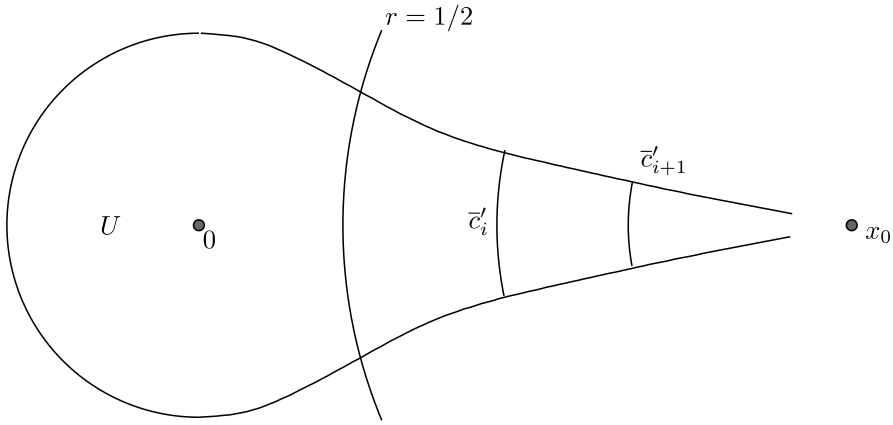

FIGURE 5. U with the metric ρg^2. (ρ-diam $c_i' \to 0$)

subset of $\{0 < r < 1/2\}$ and we can take a subsequence such that the closures $\overline{c}'_i$ are mutually disjoint. Thus we obtain a topological chain representing ξ.

In the remaining case, we may assume that c'_i converges to a point x_1 distinct from x_0. See Figure 6. We shall still use the polar coordinates (r, θ) around x_0.

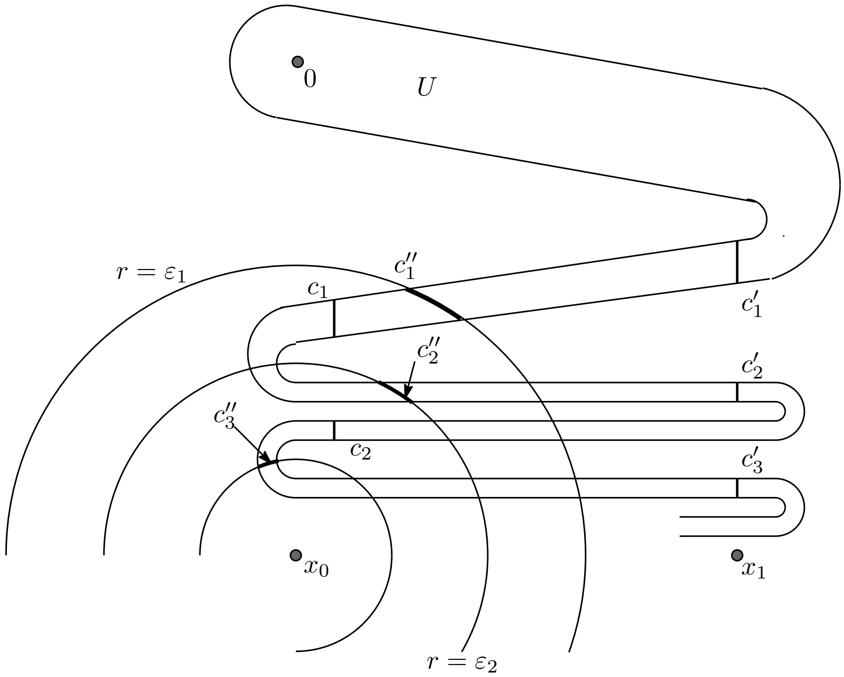

FIGURE 6

Recall that we have another chain $\{c_i\}$ converging to x_0. The chains $\{c_i\}$ has no particularly good property other than $\mathrm{diam}(c_i) \to 0$. In the worst case x_0 may belong to any $\overline{c}_i$. However passing to subsequences of $\{c'_i\}$ and $\{c_i\}$ (denoted by the same letters) and choosing a sequence of positive numbers $\epsilon_i \downarrow 0$, we may assume the following.

(1) The cross cut c_i is contained in $\{r < \epsilon_i\}$.
(2) All the c'_i is disjoint from $\{r \leq \epsilon_1\}$.
(3) The sequence $c'_1, c_1, c'_2, c_2, \cdots$ forms a chain.

Then there is a component c''_i of $\{r = \epsilon_i\} \cap U$ which separates c_i from c'_i. The chain $\{c''_i\}$ is the desired topological chain. □

A cross cut $c : \mathbb{R} \to U$ is called *extendable* if the limits $\lim_{t\to-\infty} c(t)$ and $\lim_{t\to\infty} c(t)$ exist. Then $\bar{c}$ is either a compact arc or a Jordan curve in Σ. A topological chain $\{c_i\}$ is called *extendable* if each c_i is extendable. The proof of the above lemma also shows the following lemma useful in the sequel.

LEMMA 2.3. *A prime end is represented by an extendable topological chain.*

For a hyperbolic domain U of Σ, denote by $\mathcal{P}(U)$ the set of prime ends of U. The union $\hat{U} = U \cup \mathcal{P}(U)$, topologized in a standard way, is called the *Carathéodory compactification* of U. Let us explain it in bit more details. A neighbourhood system in $\hat{U}$ of a point in U is the same as a given system in U. Choose a point $\xi \in \mathcal{P}(U)$ represented by a topological chain $\{c_i\}$. The set of points in the content $U(c_i)$, together with the prime ends represented by topological chains contained in $U(c_i)$ for each i forms a neighbourhood system of ξ.

Lemma 2.2 shows that a conformal equivalence $\phi : U \to V$ extends to a homeomorphism $\hat{\phi} : \hat{U} \to \hat{V}$. In particular $\hat{U}$ is homeomorphic to $\hat{\mathbb{D}}$ by the natural extension $\hat{\phi}$ of a Riemann mapping $\phi : U \to \mathbb{D}$, and for $\mathbb{D}$ it is clear that $\hat{\mathbb{D}}$ is homeomorphic to the closed disc $\mathbb{D} \cup \partial_\infty \mathbb{D}$. Thus $\hat{U}$ is homeomorphic to a closed disc for any hyperbolic domain U. On the other hand by the definition of topological chains, a homeomorphism f of U which extends to a homeomorphism of the closure $\overline{U}$ does extend to a homeomorphism $\hat{f}$ of the compact disc $\hat{U}$. Especially important is the rotation number of the restriction of $\hat{f}$ to $\mathcal{P}(U)$, which is called the *Carathéodory rotation number.*

A proper embedding $\gamma : [0, \infty) \to U$ is called a *ray.* A ray γ is said to *belong to* a prime end ξ if ξ is represented by a chain $\{c_i\}$ and for any i, there is $t > 0$ such that $\gamma[t, \infty) \subset U(c_i)$. The ray γ is called *extendable* if the limit $\lim_{t\to\infty} \gamma(t)$, called the *end point* of γ, exists. The end point of an extendable ray in U belongs to the frontier $\mathrm{Fr}(U)$.

A prime end ξ of U is called *extendable* if there is an extendable ray belonging to ξ. Denote by $\mathcal{EP}(U)$ the set of extendable prime ends.

LEMMA 2.4. *The end points of two extendable rays γ_i $(i = 1, 2)$ belonging to the same prime end ξ coincide.*

Proof. The end point of γ_i is the limit point of *any* topological chain representing ξ. □

Lemma 2.4 enables us to define a natural map $\Phi : \mathcal{EP}(U) \to \mathrm{Fr}(U)$.

LEMMA 2.5. *Any extendable ray belongs to some prime end.*

Proof. Given an extendable ray γ with end point $x \in \mathrm{Fr}(U)$, one can construct a topological chain from the concentric circles centered at x, by much the same argument as in the proof of Lemma 2.2. □

The above lemma says that a ray γ extendable in $U \subset \Sigma$ is extendable in the closed disc $\hat{U}$.

By an identification $\hat{\phi} : \mathcal{P}(U) \to \partial_\infty \mathbb{D}$ induced from a Riemann mapping $\phi : U \to \mathbb{D}$, the Lebesgue measure on $\partial_\infty \mathbb{D}$ is transformed to a probability measure on $\mathcal{P}(U)$. It depends upon the choice of the Riemann mapping ϕ, but its class (called *Lebesgue class*) is unique.

LEMMA 2.6. *The set $\mathcal{EP}(U)$ of extendable prime ends is conull w. r. t. the Lebesgue class. Especially $\mathcal{EP}(U)$ is dense in $\mathcal{P}(U)$.*

Proof. Let $\psi : \mathbb{D} \to U$ be the inverse Riemann mapping. Then another application of the Schwarz inequality shows

$$\int_0^{2\pi} \int_{1/2}^1 |\psi'(re^{i\theta})| r dr d\theta < \infty.$$

That is, for Lebesgue almost all θ_0, the value

$$2 \int_{1/2}^1 |\psi'(re^{i\theta_0})| dr < 4 \int_{1/2}^1 |\psi'(re^{i\theta_0})| r dr < \infty.$$

Notice that the LHS is the length of the ray $\psi\{re^{i\theta_0} \mid 1/2 \le r < 1\}$. □

REMARK 2.7. It is not the case that an extendable prime end always admits a ray of finite length. See Figure 7.

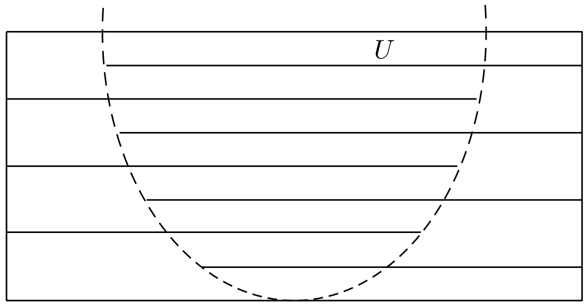

FIGURE 7

3. The Cartwright-Littlewood theorem

Let $f : \Sigma \to \Sigma$ be an orientation preserving homeomorphism which leaves a hyperbolic domain U in Σ invariant. Now f induces an orientation preserving homeomorphism on the Carathéodory compactification, $\hat{f} : \hat{U} \to \hat{U}$. The purpose of this section is to give a proof of the following theorem due to M. L. Cartwright and J. E. Littlewood ([**CL**]).

THEOREM 3.1. *Let f and U be as above. Assume that there is no fixed point in* $\mathrm{Fr}(U)$ *and that the Carathéodory rotation number of U is* 0*. Then the restriction of $\hat{f}$ to $\mathcal{P}(U)$ is Morse-Smale, and if $\xi \in \mathcal{P}(U)$ is an attractor (resp. repellor) of the restriction of $\hat{f}$ to $\mathcal{P}(U)$, then ξ is an attractor (resp. repellor) of the homeomorphism $\hat{f}$ of $\hat{U}$.*

See Figure 8. One consequence of this is the famous Cartwright-Littlewood fixed point theorem stated as Theorem 4.5 at the end of Section 4. Before giving the proof, we shall raise two examples of an invariant domain with Carathéodory rotation number 0.

EXAMPLE 3.2. There is a simple homeomorphism h of S^2 which satisfies the following conditions.

(1) The homeomorphism h preserves a continuum X.
(2) There is no periodic point in X
(3) $S^2 \setminus X$ consists of three open discs U_+, U_- and V.
(4) All the three open discs are invariant by h.
(5) The Carathéodory rotation number of V is 0.

To construct h, we start with a Morse Smale diffeomorphism g of the interval $[0, 1]$ whose fixed points are 0 and 1. Consider the suspension flow of g on the annulus $S^1 \times [0, 1]$. Define h to be the time α map of the flow, where α is any irrational number. Choose one orbit Y from $S^1 \times (0, 1)$ and let $X = S^1 \times \{0, 1\} \cup Y$ and $V = S^1 \times [0, 1] \setminus X$. Finally extend h to S^2 in an obvious way. See Figure 9. Then the homeomorphism $\hat{f}$ on the Carathéodory compactification $\hat{V}$ has two fixed prime ends.

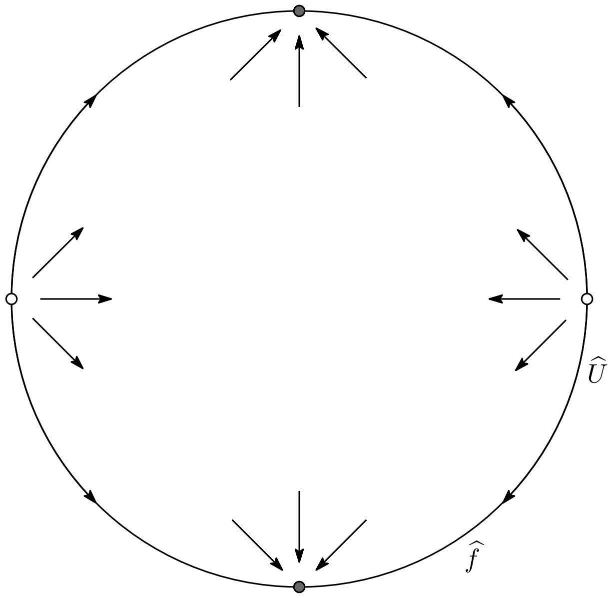

FIGURE 8

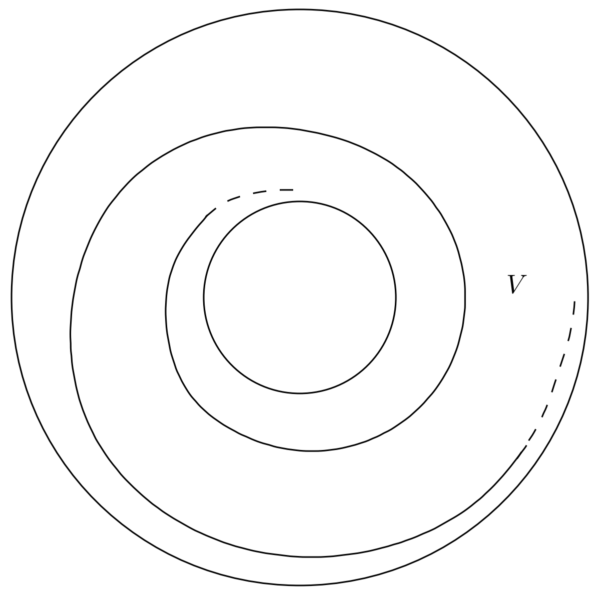

FIGURE 9

EXAMPLE 3.3. Let g be a Denjoy's C^1 diffeomorphism of S^1 whose minimal set is a Cantor set $\mathfrak{N}$. We put the suspension $T^2 = S^1 \times \mathbb{R}/(x, y) \sim (g(x), y+1)$. For an irrational number α, we define $f : T^2 \to T^2$ by $f([x, y]) = [x + \alpha, y]$. Then the minimal set of f is $\mathfrak{N} \times \mathbb{R}/\sim$. Its complement U is a simply connected invariant domain. By the same reason as in Examples 3.2, the Carathéodory rotation number of U is 0. See figure 10.

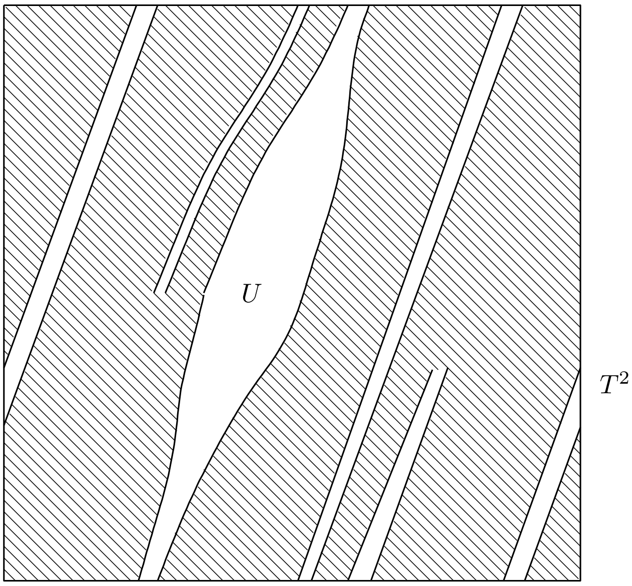

FIGURE 10

Proof of Theorem 3.1. By the assumption on the Carathéodory rotation number, the homeomorphism $\hat{f}$ has a fixed point ξ in $\mathcal{P}(U)$. Let $\{c_i\}$ be an extendable topological chain representing ξ. Recall that $\overline{c}_i$ are mutually disjoint in Σ. Also a ray contained in c_i is extendable and therefore belongs to some prime end by Lemma 2.5. This implies that the cross cut c_i is extendable in the Carathéodory compactification $\hat{U}$. The closure of c_i in $\hat{U}$ is denoted by $\hat{c}_i$. By Lemma 2.4 $\hat{c}_i$ are also mutually disjoint.

Assume for contradiction that $\hat{f}\hat{c}_i \cap \hat{c}_i \neq \emptyset$ for infinitely many i. Then again by Lemma 2.4 we have $f\overline{c}_i \cap \overline{c}_i \neq \emptyset$. Since $\mathrm{diam}(c_i) \to 0$, the point of accumulation of c_i must be a fixed point of f. Therefore we can assume $\hat{f}\hat{c}_i \cap \hat{c}_i = \emptyset$ for any i.

Let $\hat{U}(c_i)$ be the component of $\hat{U} \setminus \hat{c}_i$ not containing the base point $0 \in U$. Notice that $U(c_i) = U \cap \hat{U}(c_i)$. Then we have for each large i either $\hat{f}\hat{c}_i \subset \hat{U}(c_i)$ or $\hat{c}_i \subset \hat{f}\hat{U}(c_i)$ because ξ is a fixed point of $\hat{f}$. Assume, to fix the idea, that $\hat{f}\hat{c}_i \subset \hat{U}(c_i)$ for any i, by passing to a subsequence.

Now let N be an neighbourhood of the frontier $\mathrm{Fr}(U)$ which does not intersect the fixed point set $\mathrm{Fix}(f)$ of f. Then since $\cap_i \overline{U}(c_i) \subset \mathrm{Fr}(U)$ in Σ by Lemma 2.1, the closure of the domain $U(c_i)$ for some big i is contained in N. Fix once and for all such c_i and denote it by c. The two end points η and ζ of $\hat{c}$ form an interval $[\eta, \zeta]$ in $\mathcal{P}(U)$ containing the prime end ξ, a fixed point of $\hat{f}$. On this interval we have

$$\eta < \hat{f}\eta < \hat{f}^2\eta < \cdots < \hat{f}^2\zeta < \hat{f}\zeta < \zeta.$$

Assume that

$$\eta^\infty = \lim \hat{f}^n \eta < \zeta^\infty = \lim \hat{f}^n \zeta. \tag{3.1}$$

See Figure 11. A contradiction will show that the map $\hat{f}$ is Morse-Smale on $\mathcal{P}(U)$.

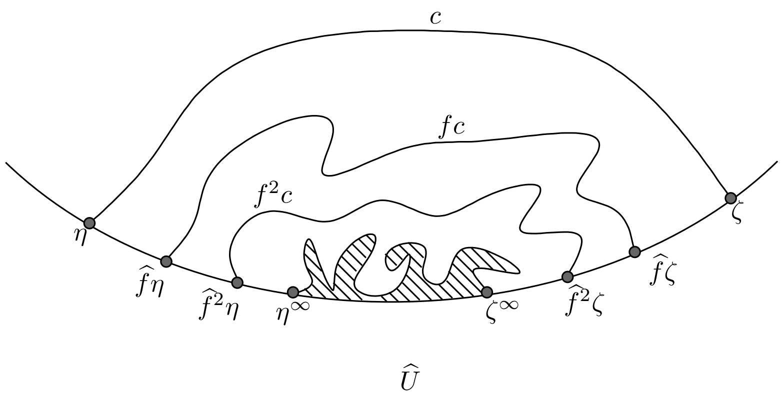

FIGURE 11. Hatched area is $U \setminus U_0$

Consider a domain

$$U_0 = U \setminus \cap_n f^n U(c)$$

and notice that $\mathrm{Fix(f)} \cap \mathrm{Fr} U_0 = \emptyset$, by the choice of c. The chain $\{f^n c\}$ of U is also a chain of U_0, and each cross cut $f^n c$ is of course extendable. An important feature of U_0 is that the intersection of the contents is empty, i. e.

$$\cap_{n=0}^\infty f^n U_0(c) = \emptyset. \tag{3.2}$$

Let us denote by $\hat{f}_0$ the homeomorphism induced by f on the Carathéodory compactification $\hat{U}_0$ of U_0. Let η_0 and ζ_0 be the prime ends in $\mathcal{P}(U_0)$ corresponding to the end points of c. Then we have

$$\eta_0 < \hat{f}_0\eta_0 < \hat{f}_0^2\eta_0 < \cdots < \hat{f}_0^2\zeta_0 < \hat{f}_0\zeta_0 < \zeta_0.$$

Let $\eta_0^\infty = \lim \hat{f}_0^n \eta_0$ and $\zeta_0^\infty = \lim \hat{f}_0^n \zeta_0$. It follows from the definition of topological chains that there is an order preserving homeomorphism between $\mathcal{P}(U) \setminus [\eta^\infty, \zeta^\infty]$ and $\mathcal{P}(U_0) \setminus [\eta_0^\infty, \zeta_0^\infty]$. Let us show $\eta_0^\infty < \zeta_0^\infty$. Assuming the contrary, we get an extendable topological chain c_i' representing $\eta_0^\infty = \zeta_0^\infty$. Let α_0^i and β_0^i be the two prime ends in $\mathcal{P}(U_0)$ corresponding to c_i'. Then clearly the sequences $\hat{f}_0^n\eta_0$ and α_0^i have the same limit $\eta_0^\infty = \zeta_0^\infty$. In other words, they are cofinal, that is, for any i, there is n such that $\alpha_0^i < \hat{f}_0^n\eta_0$ and for any n, there is i such that $\hat{f}_0^n\eta_0 < \alpha_0^i$. Likewise β_0^i and $\hat{f}_0^n\zeta$ are cofinal. Now c_i' is also an extendable topological chain of U joining α_i and β_i in $\mathcal{P}(U)$. Since $\mathcal{P}(U) \setminus [\eta^\infty, \zeta^\infty]$ and $\mathcal{P}(U_0) \setminus [\eta_0^\infty, \zeta_0^\infty]$ are order preservingly homeomorphic, we see that α_i and $\hat{f}^n\eta$ are cofinal and β_i and $\hat{f}^n\zeta$ are cofinal. Since $\{c_i'\}$ is also a topological chain of U, this shows that $\eta^\infty = \zeta^\infty$, against the assumption (3.1).

Since f is fixed point free on $\mathrm{Fr}(U_0)$ and the natural map $\Phi : \mathcal{EP}(U_0) \to \mathrm{Fr}(U_0)$ is equivariant; $\Phi \circ \hat{f}_0 = f \circ \Phi$, the set of the extendable end $\mathcal{EP}(U_0)$ is disjoint from $\mathrm{Fix}(\hat{f}_0)$. Lemma 2.6 implies that the fixed point set of $\hat{f}_0$ is nowhere dense in $\mathcal{P}(U_0)$. Thus there is a point σ in the interval $[\eta_0^\infty, \zeta_0^\infty]$ which is not fixed by $\hat{f}_0$.

See Figure 12. To fix the idea assume $\hat{f}_0\sigma > \sigma$ and let $\hat{f}_0^{-n}\sigma \downarrow \tau$. Let $\{c_i''\}$ be an

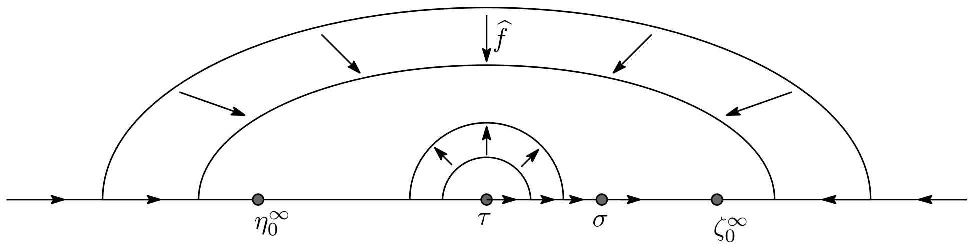

FIGURE 12

extendable topological chain of U_0 representing τ. Denote by $U_0(c_i'')$ the content of c_i'' in U_0. As before we have $\hat{f}_0(c_i'') \cap c_i'' = \emptyset$ if we pass to a subsequence. But τ is repelling on its right side. Therefore $U_0(c_i'') \subset \hat{f}_0 U_0(c_i'')$. If we choose i big enough, we have $U_0(c_i'') \subset U_0(c)$. But this is contrary to (3.2), finishing the proof that $\hat{f}$ is Morse-Smale on $\mathcal{P}(U)$.

Let us show the last part of the theorem. Assume ξ is an attractor of $\hat{f}|_{\mathcal{P}(U)}$. Choose an extendable topological chain representing ξ. Then as before we can assume $fU(c_i) \subset U(c_i)$ and $U(c_i) \cap \mathrm{Fix}(f) = \emptyset$ for any big i. Fix one such i and let $c = c_i$. Let $U_1 = U \setminus \cap_{n \geq 1} f^n U(c)$. See Figure 13. Denote the two end points

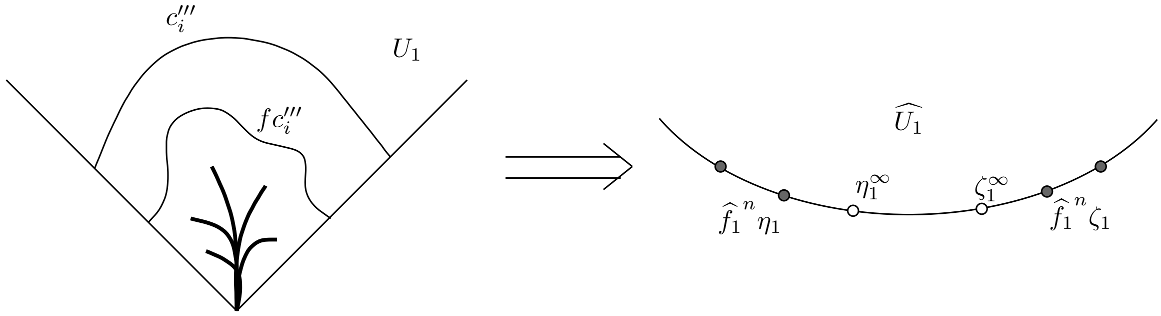

FIGURE 13. No topological chain $\{c_i'''\}$

of c in $\mathcal{P}(U_1)$ by η_1 and ζ_1 and let $\eta_1^\infty = \lim \hat{f}_1^n \eta_1$ and $\zeta_1^\infty = \lim \hat{f}_1^n \zeta_1$, where $\hat{f}_1$ is the homeomorphism of $\hat{U}_1$ induced by f. If we show that $\zeta_1^\infty \neq \eta_1^\infty$, then the same argument as before yields a contradiction. Assume $\zeta_1^\infty = \eta_1^\infty$, and take an extendable topological chain $\{c_i'''\}$ representing this prime end in $\mathcal{P}(U_1)$. It is also a topological chain for U and we have

$$U \setminus U(c_i''') = U_1 \setminus U_1(c_i''').$$

Since $\cap_i U(c_i''') = \cap_i U_1(c_i''') = \emptyset$ by Lemma 2.1, this shows $U_1 = U$. A contradiction. □

4. Minimal continuum

Let f be an orientation preserving homeomorphism of the 2-sphere S^2 which has a (nontrivial) continuum X as a minimal set. Recall that a connected component

U of $S^2 \setminus X$ is called an invariant domain if $fU = U$. The purpose of this section is to prove Theorem 1.1. We begin with the following lemma.

LEMMA 4.1. *The Carathéodory rotation number of an invariant domain U is nonzero.*

Before the proof, let us mention that Example 3.2 shows the necessity for the minimality assumption and that Example 3.3 shows that Lemma 4.1 does not hold for surfaces of nonzero genus.

Proof of Lemma 4.1. Denote by $\hat{f}$ the homeomorphism that f induces on $\hat{U}$. Assume for contradiction that the rotation number of $\hat{f}|_{\mathcal{P}(U)}$ is 0. Then the conclusion of Theorem 3.1 holds. Let α and ω be adjacent repelling and attracting fixed points on $\mathcal{P}(U)$ and choose an interval (α, ω) in $\mathcal{P}(U)$ so that $(\alpha, \omega) \cap \mathrm{Fix}(\hat{f}) = \emptyset$. By Lemma 2.6 there is a prime end $\xi \in (\alpha, \omega)$ belonging to the set $\mathcal{EP}(U)$ of the extendable prime ends near ω. Then one can choose an extendable curve $\hat{\gamma}$ joining ξ and $\hat{f}\xi$ such that $\gamma = \hat{\gamma} \cap U$ is contained in an open fundamental domain F of $\hat{f}$. (Recall that ω is an attractor of the homeomorphism $\hat{f}$.) See Figure 14. Notice

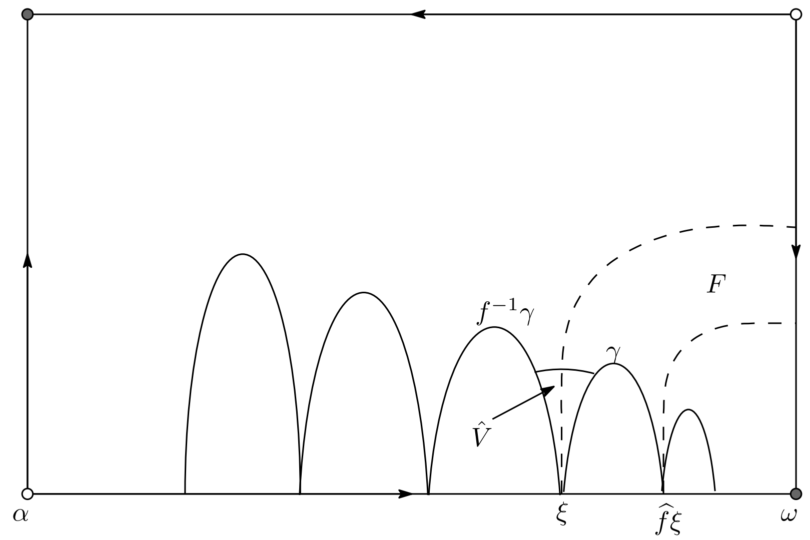

FIGURE 14. $\widehat{U}$

that the natural map $\Phi : \mathcal{EP}(U) \to X$ is equivariant; $f \circ \Phi = \Phi \circ \hat{f}$. Therefore the closure $\overline{\gamma}$ of the curve γ in S^2 joins a point, say p, with fp. Notice that $p \in X$. The cross cuts $f^n\gamma$ in U ($n \in \mathbb{Z}$) are mutually disjoint and its closure $f^n(\overline{\gamma})$ joins a point $f^n(p)$ with $f^{n+1}(p)$.

Since X is minimal and $p \in X$, there is $n > 0$ such that $f^n p$ is arbitrarily near p. Consider a small disc B centered at p such that $B \cap fB = \emptyset$. The connected component of $f^{-1}\overline{\gamma} \cup \overline{\gamma}$ that contains the point p divides B into two domains. One of it V, corresponding to $\hat{V}$ in Figure 14, is contained in U (if we choose B small enough) and the point $f^n p$ can be chosen from the component of $B \setminus (f^{-1}\overline{\gamma} \cup \overline{\gamma})$ adjacent to V. Choose a small arc δ' in B joining p with $f^n p$ which does not intersect $f^{-1}\overline{\gamma} \cup \overline{\gamma}$ except at p. Notice that $f\delta' \cap \delta' = \emptyset$. See Figure 15.

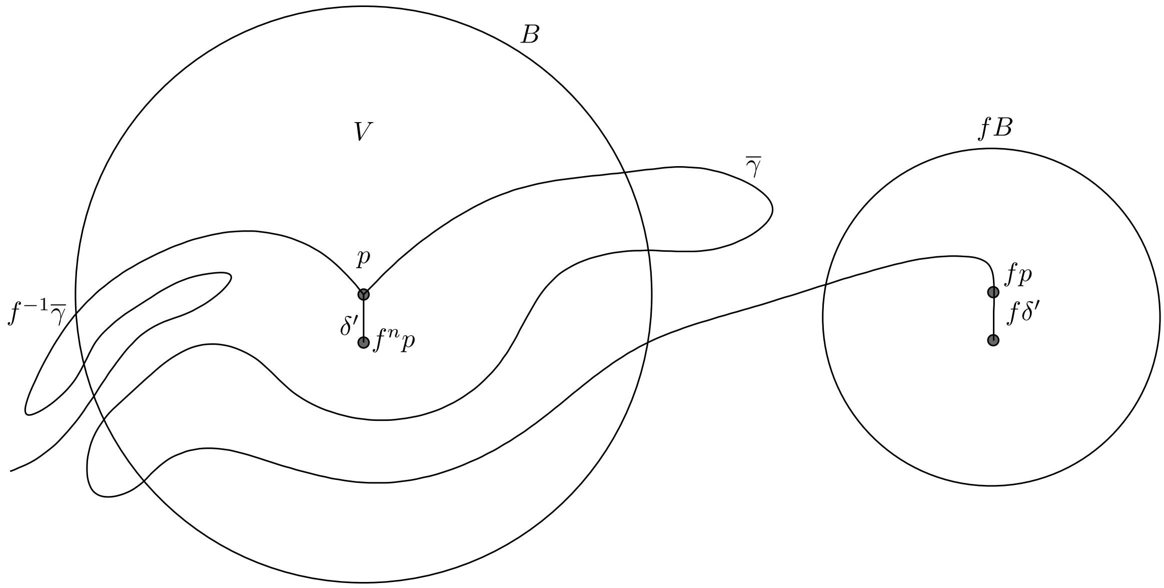

Figure 15

Consider a long simple curve $\Gamma_+ = \cup_{n\geq 0} f^n\overline{\gamma}$. Let q be the first point of intersection of $\Gamma_+ \setminus \{p\}$ with δ' (possibly $q = f^n p$) and let δ be the subarc of δ' joining p and q. Notice that q is not from $\overline{\gamma}$ since $\delta' \cap \overline{\gamma} = \{p\}$. The tiny arc δ together with the subarc Γ_+^0 of Γ_+ that joins p and q forms a Jordan curve J. See Figure 16.

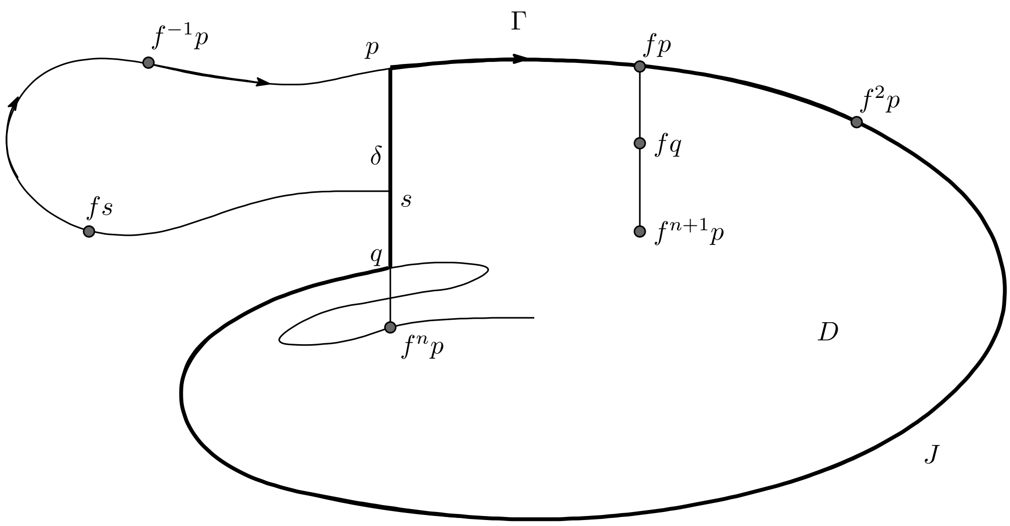

Figure 16. the curve J

Let D be the connected component of $S^2 \setminus J$ which contains fq. Then the half open arc $f\delta' \setminus \{p\}$ cannot intersect J since q is the first intersection point. Thus $f\delta' \setminus \{p\}$ and *in particular* its end point $f^{n+1}p$ is contained in D.

We also have $f^{-1}\gamma \cap \overline{D} = \emptyset$, since f^{-1} is an *orientation preserving* homeomorphism mapping a neighbourhood of fp to a neighbourhood of p. Another long curve $\Gamma_- = \cup_{n<0} f^n\overline{\gamma}$ must go arbitrarily near to the point $f^{n+1}p$ which is in D, and therefore must intersect δ. Let s be the first intersection point of $\Gamma_- \setminus \{p\}$ with δ. Then an open arc Γ_-^0 in Γ_- with end points p and s cannot intersect J and therefore $\Gamma_-^0 \cap D = \emptyset$. By the construction of δ', s is not from $f^{-1}\overline{\gamma}$ and thus $fs \in \Gamma_-^0$. On the other hand fs lies on $f\delta$ and therefore belongs to D. A contradiction. □

A closed disc D in S^2 is called *adapted* if $\partial D \cap \mathrm{Fix}(f) = \emptyset$ and $D \cup fD \neq S^2$. Given an adapted disc D, choosing the point of infinity in $S^2 \setminus (D \cup fD)$, one may consider $D \cup fD$ to be contained in $\mathbb{R}^2$. Then the degree of the map

$$id - f : \partial D \to \mathbb{R}^2 \setminus \{0\}$$

is called the *index* of f w. r. t. D and is denoted by $\mathrm{Ind_f} D$. An application of the Lefschetz index theorem yields the following lemma.

Lemma 4.2. *Let $D_1, \cdots, D_r$ be mutually disjoint adapted discs such that there is no fixed point of f in the complement of $\cup_{j=1}^r D_j$. Then we have*

$$\sum_{j=1}^{r} \mathrm{Ind_f} D_j = 2.$$

Let us return to the hypothesis of Theorem 1.1 that X is a connected minimal set of f. Given an invariant domain U, we have $\mathrm{Fix}(f) \cap U \neq \emptyset$ by Lemma 4.1 and the Brouwer fixed point theorem applied to the Carathéodory compactification $\hat{U}$.

Lemma 4.3. *The invariant domains are finite in number.*

Proof. Assume the invariant domains are infinite and denote them by U_i ($i = 1, 2, \cdots$). Choose a fixed point x_i from U_i. Then passing to a subsequence, x_i converges to a point x in S^2, which must be a fixed point of f. If x is contained in X, then X is a fixed point, which contradicts the assumption. Otherwise, U_i coincides for large i. A contradiction. □

Choose a closed disc D in U which contains $\mathrm{Fix}(f) \cap U$ in its interior. Then D is adapted and its index $\mathrm{Ind_f} D$ is independent of the choice of D. Choose one of them and denote it by $D(U)$.

Lemma 4.4. *For any invariant domain U, the index $\mathrm{Ind_f} D(U)$ is equal to 1.*

Proof. By Lemma 4.1, the Carathéodory rotation number of U is nonzero. On $\hat{U}$ the region bounded by $\partial D(U)$ and $\mathcal{P}(U)$ has no fixed point. Thus one needs only compute the index of $\hat{f}$ w. r. t. the boundary curve $\mathcal{P}(U)$. □

Now let us conclude the proof of Theorem 1.1. Lemmata 4.2, 4.3 and 4.4 clearly show that there are exactly two invariant domains.

For any $n > 1$, the minimal set X is minimal for f^n since it is connected. Applying the above result to f^n, one can show that there is no more invariant domain of f^n. Also the Carathéodory rotation number of an invariant domain must be irrational, as is shown by applying Lemma 4.1 to the iterates of f. The proof is complete.

Let us expose the Cartwright-Littlewood fixed point theorem.

THEOREM 4.5. *Let f be an orientation preserving homeomorphism of S^2. Let X be a continuum invariant by f. Assume $U = S^2 \setminus X$ is connected. Then f has a fixed point in X.*

Proof. Assume the contrary. If the Carathéodory rotation number of U is nonzero, then Lemma 4.4 shows that $\mathrm{Ind}_{\mathrm{f}} D(U) = 1$. If the rotation number is 0, Theorem 3.1 says that the homeomorphism $\hat{f}|_{\mathcal{P}(U)}$ is Morse-Smale, with $2n$ $(n \geq 1)$ fixed points. Moreover the attractors (resp. repellors) are attractors (resp. repellors) of the whole map $\hat{f}$. In this case one can compute the index just following the definition, with the result that $\mathrm{Ind}_{\mathrm{f}} D(U) = 1 - n$. Both cases contradicts Lemma 4.2. □

5. Minimal continuum with wandering domain

In [**Ha**] a pathological C^∞ diffeomorphism is constructed which has a pseudo-circle C as a minimal set. See also [**He**]. It is well known in continuum theory that there are points x in C which are not accessible from the both side. Blowing up x, as well as all the point of its orbit, we can construct a homeomorphism which has a minimal continuum with wandering domain. Conversely if there are wandering domains, one can pinch each domain to a point (see [**BNW**]).

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