Connected but not locally connected minimal sets of codimension two foliations

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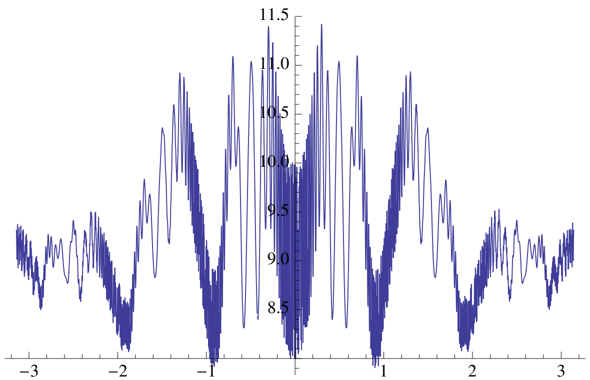
1 Purpose

Classify the topological types of minimal sets in lower dim. cases (Codim. 2)

minimal set $\overset{\text{def}}{\Longleftrightarrow}$ a closed invariant set and minimal w.r.t. the inclusion.

In this talk, the progress of my study on 'Warsaw circle with inf. many singular arcs'$\times S^1$



In ‘Foliations 2012 in Łódź’, I talked about

Theorem.
There is a C^∞ diffeomorphism of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs.

However the construction was complicated.

Recent improvement: simple construction (5 pages) and easy properties (3 pages)

Key point: ‘Warsaw circle is an inverse limit of circles’

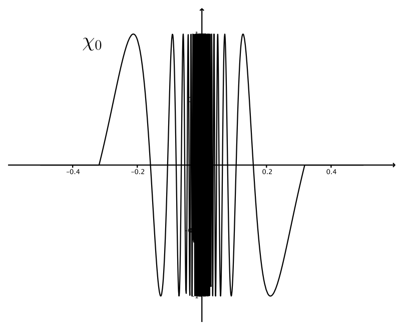
2 Warsaw circle

$S^1 = \mathbf{R}/\mathbf{Z}$

$\chi_0 : S^1 - \{0\} \to \mathbf{R}$ defined by

$$\begin{cases} \chi_0(x) = \sin \frac{1}{x} & \text{if} \quad -\frac{1}{\pi} \leq x \leq \frac{1}{\pi}, x \neq 0, \\ \chi_0(x) = 0 & \text{if} \quad \frac{1}{\pi} \leq |x| \leq \frac{1}{2} \end{cases}$$

Def 1. The closure of the graph of χ_0 ($\overline{\text{graph}\,\chi_0}$) is called the Warsaw circle X_0.



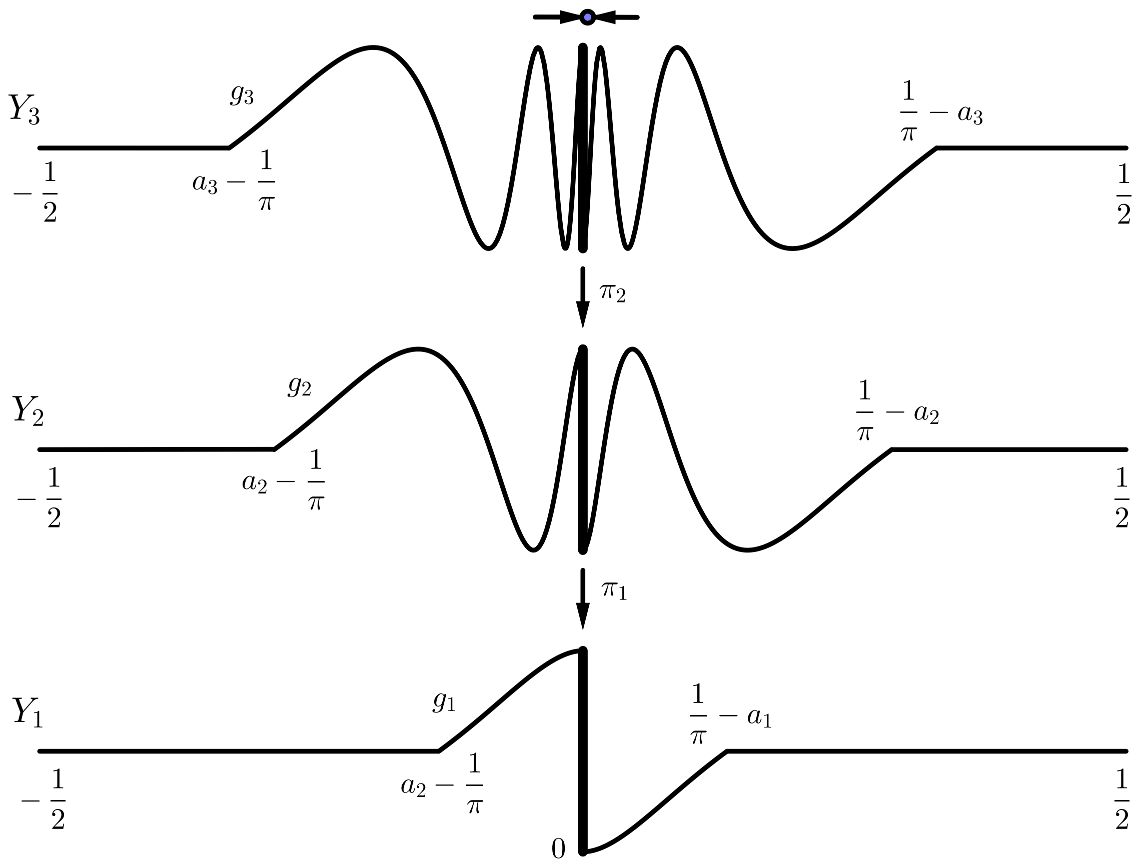
3 Warsaw circle is an inverse limit of circles

$a_n = \frac{2}{(4n-1)\pi}$ for $n = 1, 2, \cdots$

Then $\sin \frac{1}{a_n} = -1$.

We define a function $g_n : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{R}$ $(n = 1, 2, \cdots)$ by

$$g_n(x) = \begin{cases} \sin \frac{1}{x+a_n} & \text{if } 0 < x \le \frac{1}{\pi} - a_n \\ -\sin \frac{1}{-x+a_n} & \text{if } a_n - \frac{1}{\pi} \le x < 0 \\ 0 & \text{if } \frac{1}{\pi} - a_n \le |x| \le \frac{1}{2} \end{cases}$$



$$Y_n = \left(\text{graph } g_n/(-\tfrac{1}{2},0) \sim (\tfrac{1}{2},0)\right)$$

circles $\cup(\{0\} \times [-1,1])$

We define a projection $\pi_n : Y_{n+1} \to Y_n$ s.t.

$$\pi_n(x,y) = \begin{cases} (x + a_{n+1} - a_n, y) \\ \quad \text{if } a_n - a_{n+1} \leq x \leq \frac{1}{\pi} - a_{n+1} \\ (0, y) \\ \quad \text{if } a_{n+1} - a_n \leq x \leq a_n - a_{n+1} \\ (x + a_n - a_{n+1}, y) \\ \quad \text{if } -\frac{1}{\pi} + a_{n+1} \leq x \leq a_{n+1} - a_n \end{cases}$$

That is, the map π_n collapses $\{(x,y)\,;\, a_{n+1} - a_n \leq x \leq a_n - a_{n+1}\}$ into the y-axis horizontally.

For the Warsaw circle X_0, we define $h_n : X_0 \to Y_n$ s.t.

$$h_n(x,y) = \begin{cases} (x - a_n, y) & \text{if } a_n \le x \le \frac{1}{\pi} \\ (0, y) & \text{if } |x| \le a_n \\ (x + a_n, y) & \text{if } -\frac{1}{\pi} \le x \le -a_n \end{cases}$$

h_n collapses $\{(x,y)\,;\,|x| \le a_n\}$ into the y-axis horizontally.

Then $\pi_n \circ h_{n+1} = h_n$. The maps $\{h_n\}$ induce a homeomorphism from the Warsaw circle to the inverse limit (Y_n, π_n).

4 Warsaw circle with inf. many singular arcs

In order to prove the theorem, we insert infinitely many singular arcs into the circle.

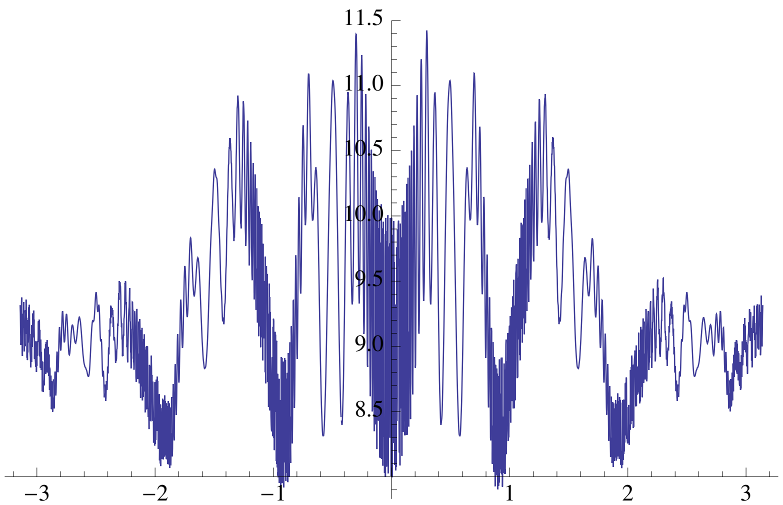
$\{b_n\}_{n\in\mathbf{Z}} > 0$: a sequence s.t. $\sum_{n\in\mathbf{Z}} b_n < \infty$.

ω: an irrational number

$\chi : S^1 - \{n\omega\,;\, n \in \mathbf{Z}\} \to \mathbf{R}$ defined by

$$\chi(x) = \sum_{n\in\mathbf{Z}} b_n \chi_0(x - n\omega).$$

Def 2. $\overline{\text{graph}\,\chi}$ in $S^1 \times \mathbf{R}$ is called a Warsaw circle with inf. many singular arcs, denoted by X.



$p_1 : S^1 \times \mathbf{R} \to S^1$; the proj. to the 1st factor. $S_n := p_1^{-1}([n\omega]) \cap X$ is called a *singular arc*, whose length is $2b_n$.

5 History

(1) Jones constructed a minimal homeomorphism of the Warsaw circle with infinitely many singular arcs X,
(introduced in the book of Gottschalk and Hedlund, 1955)

(2) Walker constructed a homeomorphism of $S^1 \times \mathbf{R}$ whose minimal set is homeomorphic to X in 1991.

(3) (Necessary conditions)

$f : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R}$; homeom

s.t. X is a minimal set.

f maps a singular fiber onto a singular fiber.

Thus we can define $\rho_f : S^1 \to S^1$ by

$$\rho_f(x) = p_1 f(x, y)$$

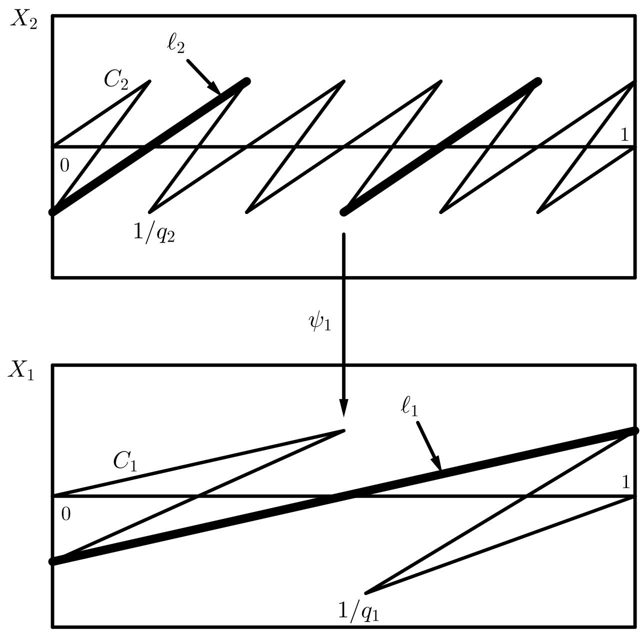
for any y satisfying $(x, y) \in X$.

Theorem. (N—, 2012)
ω: irrational number
$X = \overline{\mathrm{graph}\,\chi}$ $(\chi(x) = \sum_{n\in\mathbf{Z}} b_n \chi_0(x - n\omega))$
If b_n satisfies that
$\limsup_{n\to\infty} \frac{b_{n+1}}{b_n} \leq 1$ and $\limsup_{n\to-\infty} \frac{b_n}{b_{n+1}} \leqq 1$,
$\Longrightarrow$ there is no C^1-diffeom f of $S^1 \times \mathbf{R}$
s.t. ρ_f is a rotation and X is a minimal set.

Walker's example is not of C^1.

6 Inverse limit of circles for the construction

For the construction of our C^∞-diffeomorphism, we use an inverse limit of circles whose singular arcs are not vertical.



Preparation:

q_n: so large integers $(q_1 = 2)$

$L_n \in \mathbf{Z}_+$: $L_1 = 3$ and $L_n = \frac{2q_n}{L_1 L_2 \cdots L_{n-1}} - 1.$

$X_n = \{(x, y)\,;\, x \in \mathbf{R}/\mathbf{Z}, |y| \leq 1\}$

We define a simple closed curve

$C_n : \mathbf{R}/L_n\mathbf{Z} \to X_n$ by

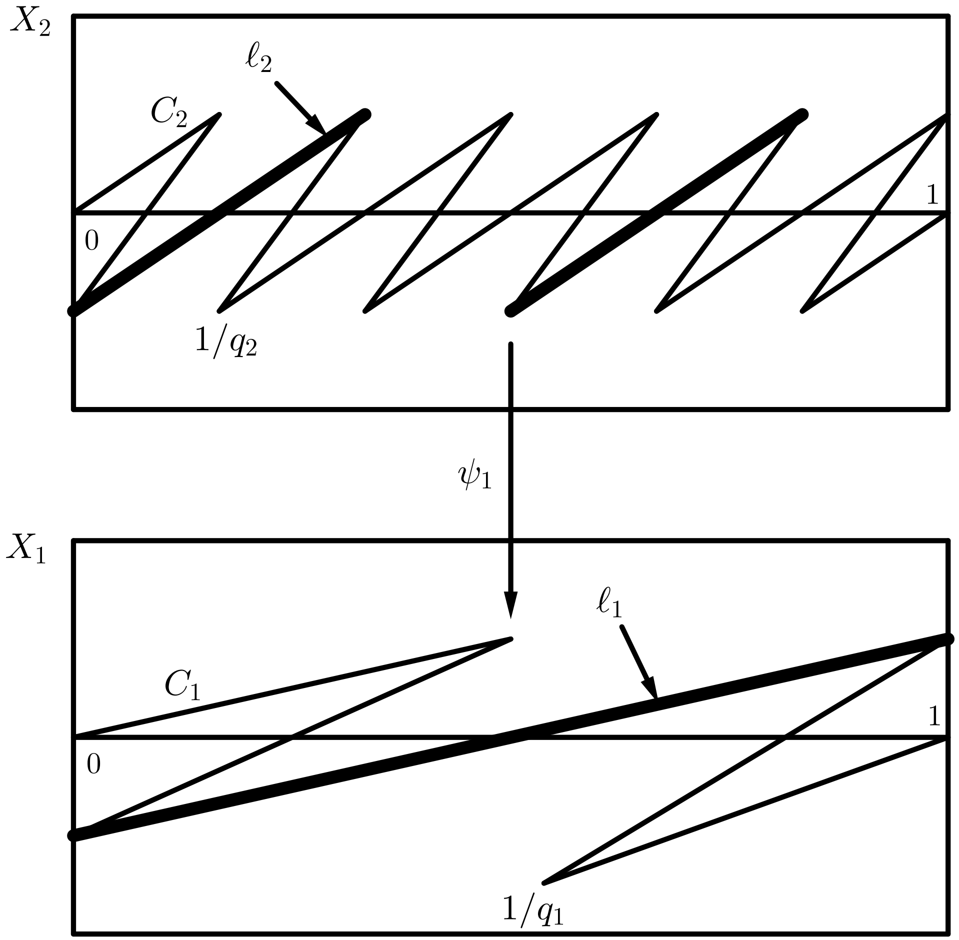
$$C_n(t) = (t, L_1 \cdots L_{n-1} t - \tfrac{1}{2})$$

$$\text{if } 0 \leq t \leq \frac{1}{L_1 \cdots L_{n-1}}$$

$$C_n(t) = (-t + \frac{2}{L_1 \cdots L_{n-1}}, \frac{L_1 \cdots L_{n-1} q_n}{q_n - L_1 \cdots L_n}(t - \frac{L_n}{q_n}) - \tfrac{1}{2})$$

$$\text{if } \frac{1}{L_1 \cdots L_{n-1}} \leq t \leq \frac{L_n}{q_n}$$

and $C_n\left(t+\frac{L_n}{q_n}\right)=R_{\frac{1}{q_n}}C_n(t)$

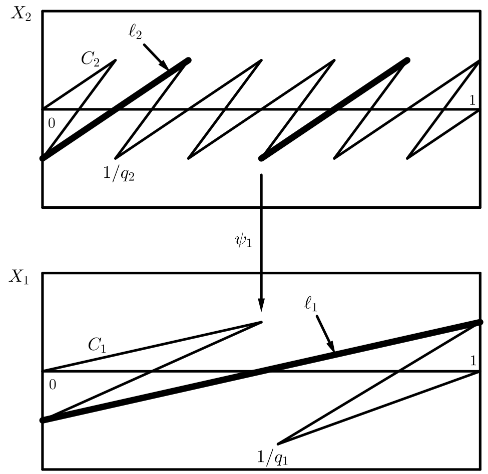


We define continuous maps

$\Psi_n : S^1 \to S^1$ by $\Psi_n(t) = p_1 C_{n+1}(L_{n+1} t)$.

$\psi_n : X_{n+1} \to X_n$ by $\psi_n(x, y) = C_n(L_n x)$

$$\begin{array}{ccc} S^1 & \xrightarrow{C_{n+1}(L_{n+1}t)} & X_{n+1} \\ \Psi_n \downarrow & \circlearrowright & \downarrow \psi_n \\ S^1 & \xrightarrow{C_n(L_n t)} & X_n \end{array}$$



We will use the inverse limit (S^1, Ψ_n) for the construction of a C^∞ diffeomorphism.

7 Overview of the construction of f

(in the manner of Handel and Fayad-Katok)

For $n = 1, 2, \cdots$, we define θ_n by $\theta_n = \sum_{i=1}^{n} \frac{1}{q_i}$.

Let R_{θ_n} denote the θ_n-rotation

$$R_{\theta_n}(x, y) = (x + \theta_n, y) \text{ in } X_n.$$

We choose a C^∞ embedding $\varphi_n : X_{n+1} \to X_n$ sufficiently near C_n satisfying

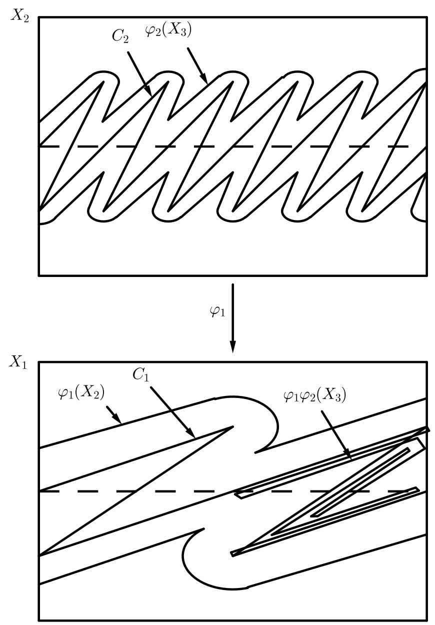
(a) $R_{\theta_n} \circ \varphi_n = \varphi_n \circ R_{\theta_n}$

(b) $\varphi_n(\ell_{n+1}) = \ell_n$ for

$$\ell_n = \{C_n(t)\, ;\, 0 \leq t \leq 1/(L_1 L_2 \cdots L_{n-1})\}.$$

(c) $\varphi_n(X_{n+1}) \subset \{(x,y) \in X_n\,;\, |y| < \frac{3}{4}\}$.

Let $\Phi_n = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$.
Then $\Phi_1(X_2) \supset \Phi_2(X_3) \supset \cdots$.
$X = \bigcap_n \Phi_n(X_{n+1})$ will be a minimal set.



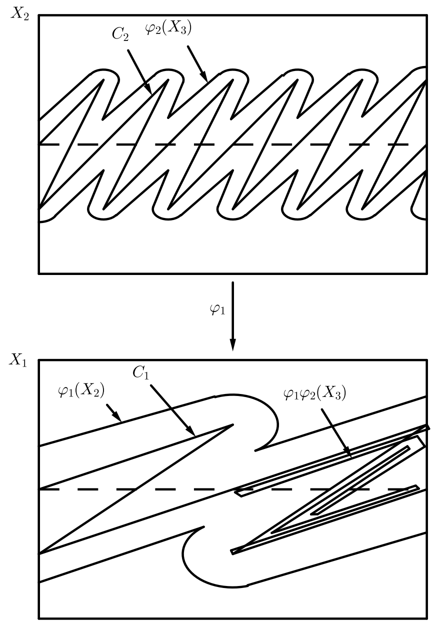
We give diffeomorphisms $f_n : X_1 \longrightarrow X_1$ satisfying

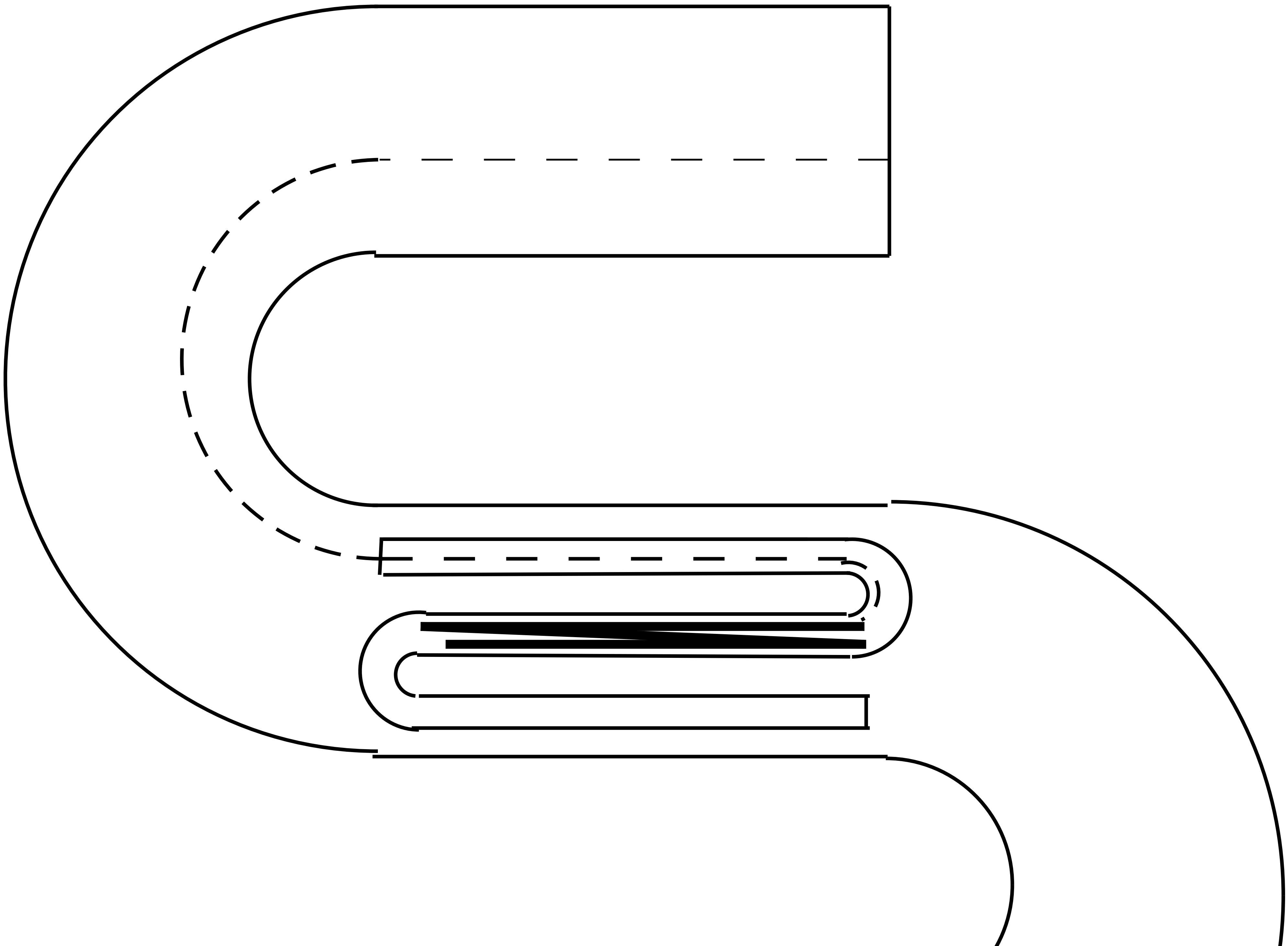
(d) $f_{n+1} = f_n$ outside $\Phi_n(X_{n+1})$ and

(e) $f_{n+1}(x, y) = \Phi_n R_{\theta_{n+1}} \Phi_n^{-1}(x, y)$
if $(x, y) \in \Phi_n(X_{n+1})$ and
$\Phi_n^{-1}(x, y) \in \{(x, y)\,; |y| \leq \frac{3}{4}\}$.

If we choose f_{n+1} sufficiently near f_n, then f_n converges to a C^∞ diffeomorphism f of X_1 as $n \longrightarrow \infty$.

Then f is a C^∞ diffeomorphism s.t. $\bigcap_n \Phi_n(X_{n+1})$ is a compact connected but not path-connected minimal set of f containing the arc ℓ_1.





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8 Minimal sets

minimal sets $\mathfrak{M}$	Is there a diffeom $f : \Sigma \to \Sigma$ whose minimal set is $\mathfrak{M}$?
(1) periodic points	C^∞
(2) circle	C^∞
(3) torus	C^∞
(4) Cantor set	C^∞ (horse shoe)
(5) Sierpiński T^2-set	C^2 by McSwiggen 1993

minimal sets	$f : \Sigma \longrightarrow \Sigma$; diffeo ?
(6) pseudo circle	C^∞ by Handel 1982
(7) "Warsaw circle"	discuss here
(8) Fayad-Katok	C^∞