Surface diffeomorphisms with connected but not path-connected minimal sets containing arcs

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Abstract

The Warsaw circle is obtained by joining the boundary of the closure of the graph of $\sin 1/x$ $(-1/\pi \leq x \leq 1/\pi)$. It is well-known as an example of a connected but not path-connected compact set. Inserting such components almost everywhere along the circle, we obtain the Warsaw circle with infinitely many singular arcs, denoted by X. In 1955, Gottschalk and Hedlund introduced in their book that Jones constructed a minimal homeomorphism on this set. However this homeomorphism is defined only on this set. In 1991, Walker first constructed a homeomorphism of $S^1 \times \mathbf{R}$ whose minimal set is homeomorphic to X. However, his homeomorphism cannot be a diffeomorphism by Theorem 1. In this paper, we will construct a C^∞ diffeomorphism of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs (Theorem 2). In order to construct a C^∞ diffeomorphism, we use the approximation by conjugation method. The key point of the construction is the fact that the Warsaw circle is an inverse limit of circles.

1 Introduction

In order to examine the dynamical properties of homeomorphisms, compact invariant sets are keys to consider the asymptotic behavior of orbits. A minimal set is a compact invariant set which is minimal with respect to the inclusion. The minimal sets play important roles as cores of compact invariant sets.

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In low dimensional dynamical systems, only few topological types of minimal sets have been found (Problem 1.6 in [3]). In this paper, we consider whether the Warsaw circle with infinitely many singular arcs (Figure 1) can be a minimal set of a surface diffeomorphism.

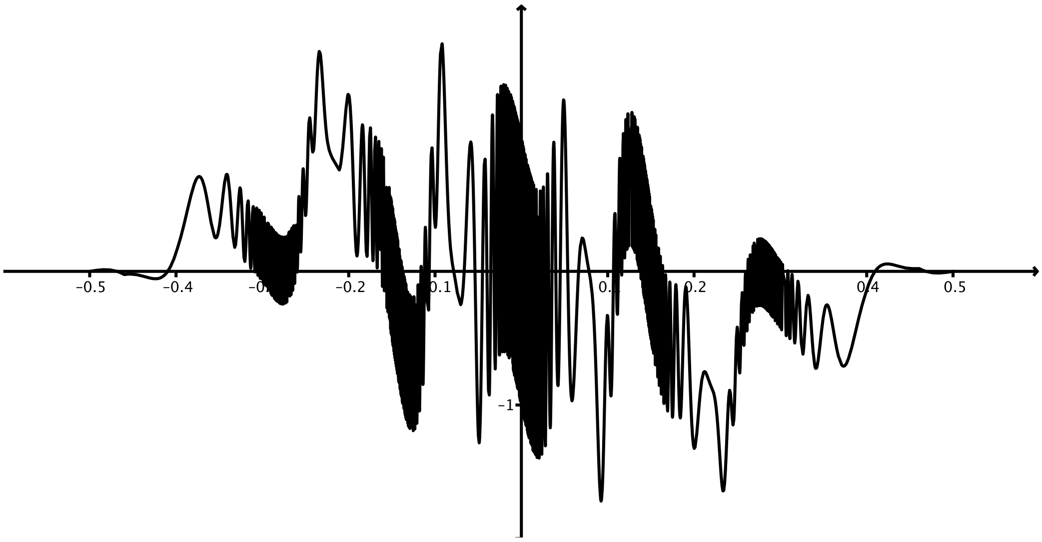

Figure 1: The Warsaw circle with infinitely many singular arcs

The Warsaw circle is the set obtained from the closure of the graph of

$$y = \sin\frac{1}{x} \quad (-1/\pi \leq x \leq 1/\pi, x \neq 0)$$

by identifying the ends. We call $\{(0, y)\,;\,|y| \leq 1\}$ a *singular arc.* The Warsaw circle is famous as an example of a connected but not path-connected set. A Warsaw circle with infinity many singular arcs is obtained by inserting infinity many such singular arcs along the circle, denoted by X (the precise definition will be given in §2).

In 1955, Gottschalk and Hedlund introduced in their book ([5]) that Jones constructed a minimal homeomorphism of X (that is, the whole set X is a minimal set). Although this set X was embedded in $S^1 \times \mathbf{R}$, the homeomorphism is defined only on the set X. In 1991, Walker ([9]) constructed a homeomorphism of $S^1 \times \mathbf{R}$ whose minimal set is homeomorphic to X. However, his homeomorphism cannot be differentiable because the singular arcs keep the vertical directions invariant and the minimality destroys the differential structure (Theorem 1).

In this paper, we will construct a C^∞ diffeomorphism of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs

(Theorem 2). In order to prove Theorem 2, we use the approximation by conjugation method (see [2] and [4]). The key point of our construction is the fact that the Warsaw circle with infinitely many singular arcs is an inverse limit of circles as the Warsaw circle is (see §4). Thus we can construct such a diffeomorphism in the same manner as Handel constructed a diffeomorphism of a surface whose minimal set is a pseudo-circle ([6]).

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2 Statement of results

First recall the homeomorphism of Gottschalk and Hedlund, which was introduced in §14 of [5] as an example communicated by Jones.

We parametrize the circle by $S^1 = \mathbf{R}/\mathbf{Z}$. Let $\chi_0 : S^1 - \{0\} \to \mathbf{R}$ denote the function defined by

$$\begin{cases} \chi_0(x) = \sin\frac{1}{x} & \text{if } -\frac{1}{\pi} \le x \le \frac{1}{\pi}, x \ne 0, \\ \chi_0(x) = 0 & \text{if } \frac{1}{\pi} \le |x| \le \frac{1}{2}. \end{cases}$$

Then the closure of the graph of χ_0 is called the *Warsaw circle*, denoted by X_0.

Let ω be an irrational number. Let $\Lambda = \{n\omega \bmod \mathbf{Z}; n \in \mathbf{Z}\}$. We choose a sequence $\{c_n\}_{n\in\mathbf{Z}}$ of positive numbers satisfying $\sum_{n\in\mathbf{Z}} c_n < \infty$. We define a function $\chi_\omega : S^1 - \Lambda \to \mathbf{R}$ by

$$\chi_\omega(x) = \sum_{n\in\mathbf{Z}} c_n \chi_0(x - n\omega).$$

For $m \in \mathbf{Z}$, the arc

$$\left\{ (m\omega, y) \in S^1 \times \mathbf{R}\,;\, -c_m \le y - \sum_{n\in\mathbf{Z}, n\ne m} c_n\chi_0(x - n\omega) \le c_m \right\}$$

is called a singular arc for $m \in \mathbf{Z}$, and the closure of the graph of χ_ω is called the *Warsaw circle with infinitely many singular arcs*, denoted by X. For $x \notin \Lambda$, χ_ω is continuous at x ([5]). Thus X is the union of the graph of χ_ω and singular arcs S_m $(m \in \mathbf{Z})$.

The rotation by ω on S^1 induces a homeomorphism on graph χ_ω. By [5], this homeomorphism is uniformly continuous on graph χ_ω, and thus it can be extended on the closure of the graph of χ_ω. This is the minimal homeomorphism of X introduced in [5].

On the other hand, we assume that f is a homeomorphism of $S^1 \times \mathbf{R}$ such that X is a minimal set of f. Then f maps each singular arc onto a singular arc. Let p_i $(i = 1, 2)$ denote the projection to the i-th factor of $S^1 \times \mathbf{R}$. Then we can define an induced homeomorphism $\rho_f : S^1 \to S^1$ by $\rho_f(x) = p_1 f(x, y)$ for any $(x, y) \in X$.

Theorem 1. *Let ω be an irrational number. Let $\{c_n\}_{n \in \mathbf{Z}}$ be a sequence of positive numbers satisfying $\sum_{n \in \mathbf{Z}} c_n < \infty$. Let X denote the closure of the graph of χ_ω. If c_n satisfies that $\limsup_{n \to \infty} \frac{c_{n+1}}{c_n} \leq 1$ and $\limsup_{n \to -\infty} \frac{c_n}{c_{n+1}} \leq 1$, then there is no C^1-diffeomorphism f of $S^1 \times \mathbf{R}$ such that the induced homeomorphism ρ_f of S^1 is a rotation and X is a minimal set of f.*

For the homeomorphism f constructed by Walker, ρ_f is a rotation and $c_n = \frac{1}{2^{|n|}}$. Thus this cannot be of class C^1 by Theorem 1.

Theorem 2. *There is a C^∞ diffeomorphism f of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs.*

3 Necessary conditions

In this section, we will prove Theorem 1. We assume that there is a C^1 diffeomorphism f of $S^1 \times \mathbf{R}$ such that the induced homeomorphism ρ_f of S^1 is a rotation and $X = \overline{\mathrm{graph}\,\chi_\omega}$ is a minimal set for an irrational number ω. In the following, we will deduce the contradiction.

Let $\Omega_+ = \{(x, y)\,;\, y > y_0 \text{ for any } (x, y_0) \in X\}$. Since $S^1 \times \mathbf{R} - X$ consists of two connected open sets, Ω_+ is invariant under f or f^2.

Proposition 1. *X is a minimal set of f^2.*

Proof. Suppose that there is a compact subset C of X invariant under f^2. Then $C \cup f(C)$ is invariant under f, and thus $C \cup f(C) = X$. Since $C \cap f(C)$ is also invariant under f, either $X = C$ or $C \cap f(C) = \emptyset$ holds. Now X is connected. Thus $C \cap f(C)$ is not empty, and thus $X = C$. Therefore X is a minimal set of f^2. ∎

Thus we have only to prove Theorem 1 when $f(\Omega_+) = \Omega_+$.

Proof of Theorem 1. Let $S_n = p_1^{-1}(n\omega) \cap X$. Since $f(S_0)$ is a singular arc, there is $n_0 \in \mathbf{Z}$ such that $f(S_0) = S_{n_0}$. Thus $\rho_f(0) = n_0\omega$. We choose a universal covering $\widetilde{\rho}_f$ of ρ_f so that $\widetilde{\rho}_f(0) = n_0\omega$. Then $\widetilde{\rho}_f(x) = x + n_0\omega$ for any $x \in \mathbf{R}$ because ρ_f is a rotation. As a consequence, $f(S_i) = S_{i+n_0}$ for any $i \in \mathbf{Z}$.

We assume that $n_0 > 0$. We can prove the other case similarly.

Let (x, y) be a point of X such that $p_1^{-1}(x) \cap X$ consists of one point, i. e. $x \notin \Lambda$. We take an arbitrary $\varepsilon > 0$ and an arbitrary neighborhood W of (x, y) in $S^1 \times \mathbf{R}$. Since $p_1^{-1}(x) \cap X$ consists of one point, there is a neighborhood U of x in S^1 such that $p_1^{-1}(U) \cap X$ is contained in W. Since $\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} \le 1$, there is $I > 0$ such that $\frac{c_{i+1}}{c_i} < \sqrt[n_0]{1+\varepsilon}$ for any $i \ge I$. We choose an integer i_0 greater than or equal to I such that $i_0\omega \in U$. Then S_{i_0} is contained in W. By the mean value theorem, there is z_{i_0} of S_{i_0} such that $\frac{\partial p_2 \circ f}{\partial y}(z_{i_0}) = \frac{c_{i_0+n_0}}{c_{i_0}}$. Now

$$\frac{c_{i_0+n_0}}{c_{i_0}} = \frac{c_{i_0+1}}{c_{i_0}} \frac{c_{i_0+2}}{c_{i_0+1}} \cdots \frac{c_{i_0+n_0}}{c_{i_0+n_0-1}} < 1 + \varepsilon.$$

Thus we conclude that, for any ε and neighborhood W of (x, y), there is a point in W such that $\frac{\partial p_2 \circ f}{\partial y} < 1 + \varepsilon$. Since ε and W can be chosen so small, we obtain $\frac{\partial p_2 \circ f}{\partial y}(x, y) \le 1$.

The set $\{(x, y)\,;\, p_1^{-1}(x) \cap X \text{ consists of one point}\}$ is dense in X. Thus $\frac{\partial p_2 \circ f}{\partial y}$ is less than or equal to 1 on the whole X. Now $f(S_i) = S_{i+n_0}$ for any $i \in \mathbf{Z}$. Thus we obtain $\cdots \ge c_{-2n_0} \ge c_{-n_0} \ge c_0 \ge \cdots$. However this contradicts the assumption $\sum_{n\in\mathbf{Z}} c_n < \infty$. ∎

4 C^∞ construction

4.1 The Warsaw circle is an inverse limit of circles.

We start from showing that the Warsaw circle is an inverse limit of circles: Let $a_n = \frac{2}{(4n-1)\pi}$ for $n = 1, 2, \cdots$. Then $\sin \frac{1}{a_n} = -1$. We define a function $g_n : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{R}$ $(n = 1, 2, \cdots)$ by

$$g_n(x) = \begin{cases} \sin \frac{1}{x+a_n} & \text{if } 0 < x \le \frac{1}{\pi} - a_n \\ -\sin \frac{1}{-x+a_n} & \text{if } a_n - \frac{1}{\pi} \le x < 0 \\ 0 & \text{if } \frac{1}{\pi} - a_n \le |x| \le \frac{1}{2} \text{ or } x = 0 \end{cases}$$

(see Figure 2). Let Y_n denote the set obtained by first identifying the end points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ of the graph of g_n and then adding the interval $\{0\} \times [-1, 1]$. Then Y_n is homeomorphic to the circle. We define a projection

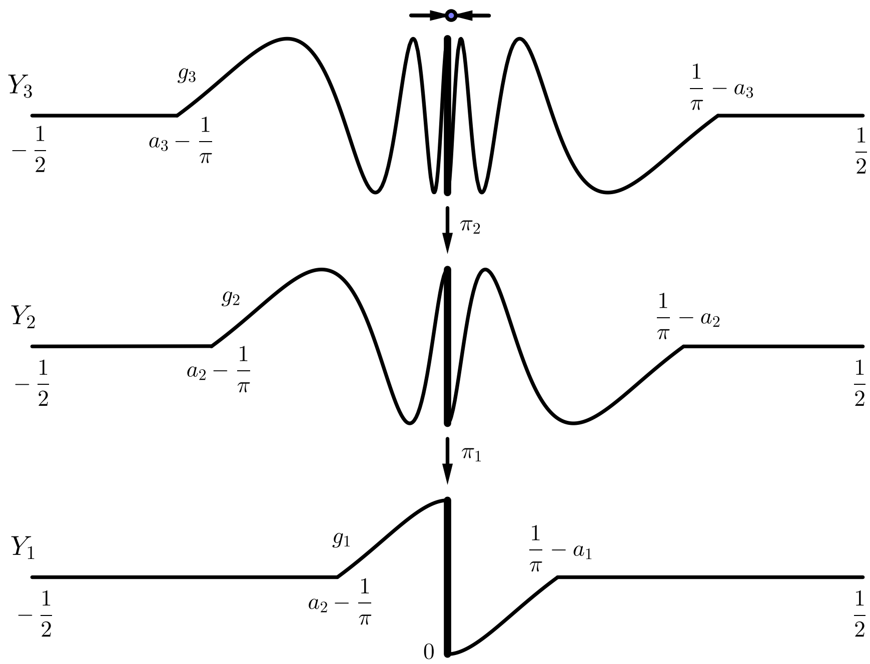

Figure 2: The Warsaw circle is a circle inverse limit

$\pi_n : Y_{n+1} \longrightarrow Y_n$ by

$$\pi_n(x,y) = \begin{cases} \left(\frac{1/2-1/\pi+a_n}{1/2-1/\pi+a_{n+1}}(x-\frac{1}{2})+\frac{1}{2}, 0\right) & \text{if } \frac{1}{\pi} - a_{n+1} \leq x \leq \frac{1}{2} \\ (x + a_{n+1} - a_n, y) & \text{if } a_n - a_{n+1} \leq x \leq \frac{1}{\pi} - a_{n+1} \\ (0, y) & \text{if } a_{n+1} - a_n \leq x \leq a_n - a_{n+1} \\ (x + a_n - a_{n+1}, y) & \text{if } -\frac{1}{\pi} + a_{n+1} \leq x \leq a_{n+1} - a_n \\ \left(\frac{1/2-1/\pi+a_n}{1/2-1/\pi+a_{n+1}}(x+\frac{1}{2})-\frac{1}{2}, 0\right) & \text{if } -\frac{1}{2} \leq x \leq -\frac{1}{\pi} + a_{n+1}. \end{cases}$$

That is, the map π_n collapses the subset $\{(x, y)\,;\, a_{n+1} - a_n \leq x \leq a_n - a_{n+1}\}$ into the y-axis horizontally.

For the Warsaw circle X_0, we define $h_n : X_0 \to Y_n$ by

$$h_n(x,y) = \begin{cases} \left(\frac{1/\pi-a_n-1/2}{1/\pi-1/2}(x-\frac{1}{2})+\frac{1}{2}, y\right) & \text{if } \frac{1}{\pi} \leq x \leq \frac{1}{2} \\ (x - a_n, y) & \text{if } a_n \leq x \leq \frac{1}{\pi} \\ (0, y) & \text{if } |x| \leq a_n \\ (x + a_n, y) & \text{if } -\frac{1}{\pi} \leq x \leq -a_n \\ \left(\frac{-1/\pi+a_n+1/2}{-1/\pi+1/2}(x+\frac{1}{2})-\frac{1}{2}, y\right) & \text{if } -\frac{1}{2} \leq x \leq -\frac{1}{\pi}. \end{cases}$$

Then $\pi_n \circ h_{n+1} = h_n$. The maps $\{h_n\}$ induce a homeomorphism from the Warsaw circle to the inverse limit (Y_n, π_n).

4.2 Inverse limit of circles for the construction.

Inverse limits of circles are adequate to make minimal dynamical systems. However the above structure is not convenient for our purpose by Theorem 1. Thus we introduce an inverse limit of circles whose singular arcs are not vertical for the Warsaw circle with infinitely many singular arcs in order to prove Theorem 2.

Let $q_1 = 2$. We choose large positive integers q_n $(n = 1, 2, \cdots)$ inductively. Let L_n denote the positive numbers defined by $L_1 = 3$ and

$$L_n = q_n \left(\frac{2}{L_1 L_2 \cdots L_{n-1}} - \frac{1}{q_n} \right)$$

for $n > 1$. Although we need several conditions on q_n for our construction, we only assume here that $q_n = k_n q_{n-1} L_1 L_2 \cdots L_{n-1}$ for some $k_n \in \mathbf{Z}_+$. In particular, q_n is a multiple of q_{n-1} and L_n is an integer. Let $X_n = \{(x, y)\,;\, x \in \mathbf{R}/\mathbf{Z}, |y| \leq 1\}$ and let p_i denote the i-th projection of X_n $(i = 1, 2)$. Let R_θ denote the θ-rotation $R_\theta(x, y) = (x + \theta, y)$ in X_n. We define a simple closed curve $C_n : \mathbf{R}/L_n\mathbf{Z} \to X_n$ by

$$C_n(t) = \begin{cases} (t, L_1 \cdots L_{n-1} t - \frac{1}{2}) & \text{if } 0 \leq t \leq \frac{1}{L_1 \cdots L_{n-1}} \\ \left(-t + \frac{2}{L_1 \cdots L_{n-1}}, \frac{L_1 \cdots L_{n-1} q_n}{q_n - L_1 \cdots L_n}(t - \frac{L_n}{q_n}) - \frac{1}{2}\right) & \text{if } \frac{1}{L_1 \cdots L_{n-1}} \leq t \leq \frac{L_n}{q_n} \end{cases}$$

and $C_n\left(t + \frac{L_n}{q_n}\right) = R_{\frac{1}{q_n}} C_n(t)$ (see Figure 3).

Then $C_n(0) = (0, -\frac{1}{2})$, $C_n(\frac{1}{L_1 \cdots L_{n-1}}) = (\frac{1}{L_1 \cdots L_{n-1}}, \frac{1}{2})$, $C_n(\frac{L_n}{q_n}) = (\frac{1}{q_n}, -\frac{1}{2})$ and C_n connects these points by line segments. Let $\ell_n = \{C_n(t)\,;\, 0 \leq t \leq \frac{1}{L_1 L_2 \cdots L_{n-1}}\}$. The slope of ℓ_n is $L_1 \cdots L_{n-1}$, which tends to ∞ very fast as $n \to \infty$. Furthermore, the curve C_n is invariant under $R_{\frac{1}{q_n}}$ and is contained in $\mathbf{R}/\mathbf{Z} \times [-1/2, 1/2]$. If we further assume that $q_n < L_1 L_2 \cdots L_n$, then we obtain

$$0 < \frac{1}{L_1 \cdots L_n} < \frac{1}{q_n} < \frac{1}{L_1 \cdots L_n} + \frac{1}{q_n} < \frac{2}{q_n} < \cdots < 1.$$

Let $\psi_n : X_{n+1} \to X_n$ denote the map defined by $\psi_n(x, y) = C_n(L_n x)$. Notice that $\psi_n(\ell_{n+1}) = \ell_n$ and ψ_n commutes with $R_{\frac{1}{q_n}}$. The latter implies that ψ_n commutes with R_{θ_n} since θ_n is a multiple of $\frac{1}{q_n}$.

We define a continuous map $\Psi_n : S^1 \to S^1$ by $\Psi_n(t) = p_1 C_{n+1}(L_{n+1} t)$. Then $\psi_n(C_{n+1}(L_{n+1} t)) = C_n(L_n \Psi_n(t))$ because, for $(x, y) = C_{n+1}(L_{n+1} t)$,

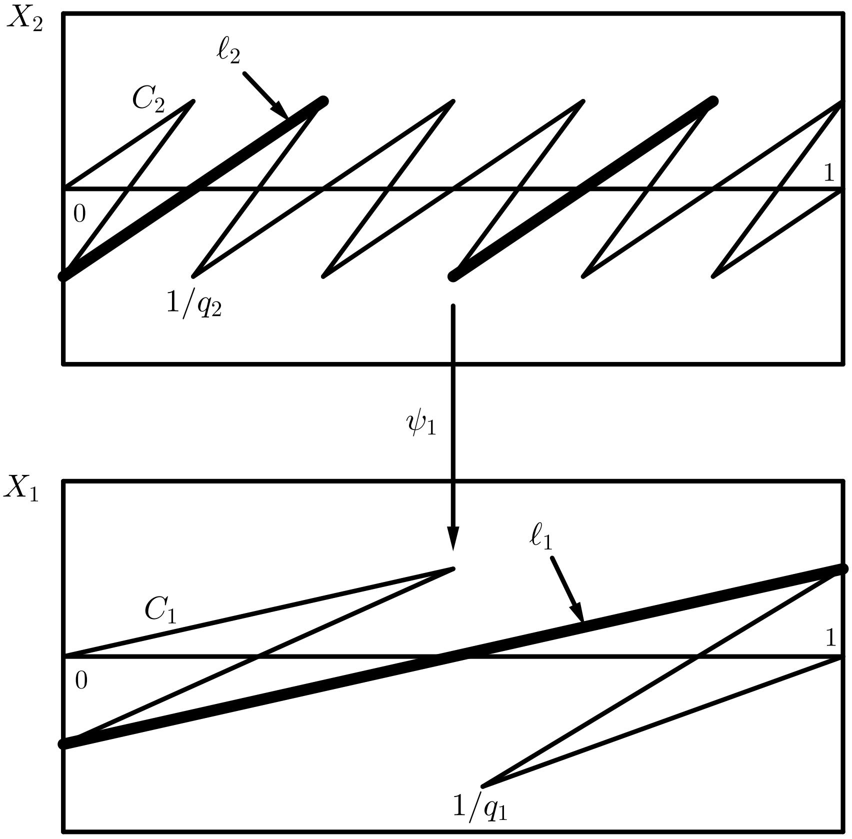

Figure 3: Circles C_n

$\psi_n(x, y) = C_n(L_n x) = C_n(L_n \Psi_n(t))$. Thus the following diagram commutes.

$$\begin{array}{ccc} S^1 & \xrightarrow{C_{n+1}(L_{n+1}t)} & X_{n+1} \\ \Psi_n \downarrow & \circlearrowleft & \downarrow \psi_n \\ S^1 & \xrightarrow{C_n(L_n t)} & X_n \end{array}$$

We will use the inverse limit (S^1, Ψ_n) as a core for the construction of a C^∞ diffeomorphism in Theorem 2 (see [1]).

4.3 Overview of the construction.

We give an angle $\theta_n \in \mathbf{R}/\mathbf{Z}$ by $\theta_n = \sum_{i=1}^n \frac{1}{q_i}$ for $n = 1, 2, \cdots$. We choose a C^∞ embedding $\varphi_n : X_{n+1} \to X_n$ sufficiently near ψ_n satisfying

(a) $R_{\theta_n} \circ \varphi_n = \varphi_n \circ R_{\theta_n}$,

(b) $\varphi_n(\ell_{n+1}) = \ell_n$,

(c) $\varphi_n(X_{n+1}) \subset \{(x, y) \in X_n\,;\, |y| < \frac{3}{4}\}$.

Let $\Phi_n = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$. Then $\Phi_1(X_2) \supset \Phi_2(X_3) \supset \cdots$. We will give a diffeomorphism $f_n : X_1 \to X_1$ satisfying

(d) $f_{n+1} = f_n$ outside $\Phi_n(X_{n+1})$ and

(e) $f_{n+1}(x, y) = \Phi_n R_{\theta_{n+1}} \Phi_n^{-1}(x, y)$ if $(x, y) \in \Phi_n(X_{n+1})$ and $\Phi_n^{-1}(x, y) \in \{(x, y)\,;|y| \le \frac{3}{4}\}$.

If we choose f_{n+1} sufficiently near f_n, then we can show that f_n converges to a C^∞ diffeomorphism f of X_1 as $n \to \infty$. The proof is based on the comparison of f_n and f_{n+1} in the middle part. Thanks to the condition $R_{\theta_n} \circ \varphi_n = \varphi_n \circ R_{\theta_n}$, the equation $f_{n-1} = \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1}$ can be written as $\Phi_n R_{\theta_n} \Phi_n^{-1}$, while $f_n = \Phi_n R_{\theta_{n+1}} \Phi_n^{-1}$. The crucial point is that we can choose the number q_{n+1} after the construction of Φ_n. Letting $|\theta_{n+1} - \theta_n|$ small enough compared with Φ_n, we get the desired convergence.

In the following, we will give the precise construction of f and will show that $\bigcap_n \Phi_n(X_{n+1})$ is a connected but not path-connected minimal set of f containing the arc ℓ_1.

4.4 Precise construction.

Let $X_n = \{(x, y)\,;\, x \in \mathbf{R}/\mathbf{Z}, |y| \le 1\}$ for $n = 1, 2, \cdots$. Let d denote the metric of X_n induced from the Euclidean metric, and let diam F denote the diameter of a set F. We define the rotation $R_\theta : X_n \to X_n$ by $R_\theta(x, y) = (x + \theta, y)$.

Let $q_1 = 2$ and $\theta_1 = \frac{1}{q_1}$. We define $f_1 : X_1 \to X_1$ by $f_1(x, y) = R_{\theta_1}(x, y)$ for $x \in \mathbf{R}/\mathbf{Z}$ and $|y| \le 1$. Let $L_1 = 3$. We define a simple closed curve $C_1 : \mathbf{R}/L_1\mathbf{Z} \to X_1$ by

$$C_1(t) = \begin{cases} (t, t - \frac{1}{2}) & \text{if } 0 \le t \le 1 \\ (-t + 2, \frac{q_1}{q_1 - L_1}\left(t - \frac{L_1}{q_1}\right) - \frac{1}{2}) & \text{if } 1 \le t \le \frac{L_1}{q_1} \end{cases}$$

and $C_1(t + \frac{L_1}{q_1}) = R_{1/q_1} C_1(t)$ for any $t \in \mathbf{R}/L_1\mathbf{Z}$ (see Figure 3). Then L_1 is the length of $p_1 \circ C_1$ and C_1 is invariant under R_{1/q_1}. Let ℓ_1 denote the segment $\{(t, t - \frac{1}{2})\,;\, 0 \le t \le 1\}$.

We define $\psi_1 : X_2 \to X_1$ by $\psi_1(x, y) = C_1(L_1 x)$. Then $\psi_1 \circ R_{1/q_1} = R_{1/q_1} \circ \psi_1$. Let $\ell_2 = \{(t, L_1 t - \frac{1}{2})\,;\, 0 \le t \le \frac{1}{L_1}\}$. Then ψ_1 maps ℓ_2 onto ℓ_1. We choose a C^∞ embedding $\varphi_1 : X_2 \to X_1$ along the curve C_1 satisfying

(1) $d(\varphi_1(x,y), \psi_1(x,y)) < 1/16$ for any $(x,y) \in X_2$. In particular,

$$\mathrm{diam}\{(x,y)\,;\,\varphi_1(x,y)\,;\,|y| \le 1\} < \frac{1}{8}$$

because $\{\psi_1(x,y)\,;\,|y|\le 1\}$ consists of one points.

(2) $\varphi_1 \circ R_{1/q_1} = R_{1/q_1} \circ \varphi_1$ (i. e. $\varphi_1 \circ R_{\theta_1} = R_{\theta_1} \circ \varphi_1$).

(3) $\varphi_1|\ell_2 = \psi_1|\ell_2$. In particular, $\varphi_1(\ell_2) = \ell_1$, $\varphi_1(0, -\frac{1}{2}) = (0, -\frac{1}{2})$ and $\varphi_1(\frac{1}{L_1}, \frac{1}{2}) = (1, \frac{1}{2})$.

(4) $\varphi_1(X_2) \subset \{(x,y)\,;\,|y| < \frac{3}{4}\}$. Here we remark that (1) implies (4) because $|p_2\psi_1(z)| \le \frac{1}{2}$ for $z \in X_2$.

Next we choose a large integer q_2 satisfying that

(5) There is $k_2 \in \mathbf{Z}_+ = \{n \in \mathbf{Z}\,;\,n > 0\}$ such that $q_2 = k_2 q_1 L_1$.

(6) If $z_1, z_2 \in X_2$ and $d(z_1, z_2) \le 2/q_2$, then $d(\varphi_1(z_1), \varphi_1(z_2)) < 1/4$.

(7) For $\theta_2 = \frac{1}{q_1} + \frac{1}{q_2}$, $q_2\theta_2$ and q_2 are relatively prime. For example, if $q_2 = kq_1^2$ for some $k \in \mathbf{Z}_+$, then $q_2\theta_2 = 1 + kq_1$ and $q_2 = kq_1^2$ are relatively prime.

Here we remark that φ_1 is determined independent of the choice of q_2.

We choose a smooth increasing function $\eta_2 : [\frac{3}{4}, 1] \to \mathbf{R}$ so that $\eta_2(3/4) = \theta_2$, $\eta_2(1) = \theta_1$ and η_2 is constant on neighborhoods of $3/4$ and 1. We define a C^∞ diffeomorphism $f_2 : X_1 \to X_1$ by

$$f_2(x,y) = \begin{cases} f_1(x,y) & \text{outside } \varphi_1(X_2) \\ \varphi_1 R_{\eta_2(|t|)} \varphi_1^{-1}(x,y) & \text{if } (x,y) \in \varphi_1(X_2) \text{ and } \frac{3}{4} \le |p_2\varphi_1^{-1}(x,y)| \le 1 \\ \varphi_1 R_{\theta_2} \varphi_1^{-1}(x,y) & \text{if } (x,y) \in \varphi_1(X_2) \text{ and } |p_2\varphi_1^{-1}(x,y)| \le \frac{3}{4}, \end{cases}$$

where f_2 is well defined by (2). If q_2 is large enough, then f_2 is assumed to be $\frac{1}{4}$-closed to f_1 in C^2-topology. Let $L_2 = q_2\left(\frac{2}{L_1} - \frac{1}{q_2}\right) \in \mathbf{Z}$. Then $\frac{1}{L_1 L_2} = \frac{1}{2q_2 - L_1} < \frac{1}{q_2}$ because $q_2 > L_1$. We define a simple closed curve $C_2 : \mathbf{R}/L_2\mathbf{Z} \to X_2$ by

$$C_2(t) = \begin{cases} (t, L_1 t - \frac{1}{2}) & \text{if } 0 \le t \le \frac{1}{L_1} \\ \left(-t + \frac{2}{L_1}, \frac{L_1 q_2}{q_2 - L_1 L_2}(t - \frac{L_2}{q_2}) - \frac{1}{2}\right) & \text{if } \frac{1}{L_1} \le t \le \frac{L_2}{q_2} \end{cases}$$

and $C_2(t + \frac{L_2}{q_2}) = R_{\frac{1}{q_2}} C_2(t)$ for any t. Then C_2 is invariant under R_{1/q_2}.

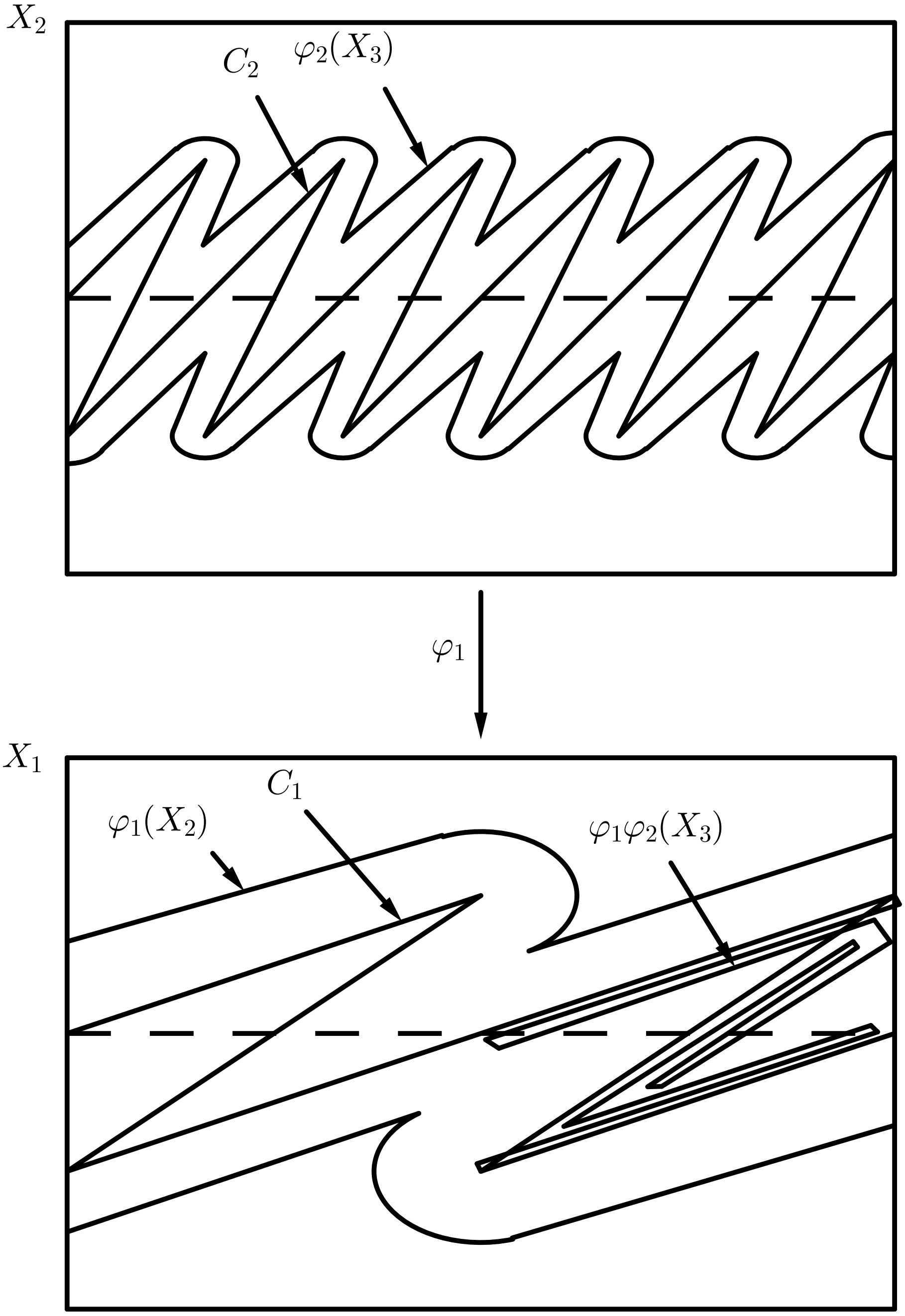

Figure 4: $\varphi_1 : X_2 \to X_1$

We define φ_n and f_n inductively as follows: We assume that f_i, C_i, L_i, q_i, ℓ_i, θ_i $(i = 1, 2, \cdots, n-1)$ and φ_i, ψ_i $(i = 1, 2, \cdots, n-2)$ satisfying the following conditions have already been given:

There is $k_{n-1} \in \mathbf{Z}_+$ such that $q_{n-1} = k_{n-1} q_{n-2} L_1 \cdots L_{n-2}$.
$L_{n-1} = q_{n-1}\left(\frac{2}{L_1 L_2 \cdots L_{n-2}} - \frac{1}{q_{n-1}}\right) \in \mathbf{Z}$.
$C_{n-1} : \mathbf{R}/L_{n-1}\mathbf{Z} \to X_{n-1}$

$$\begin{cases} C_{n-1}(t) = \begin{cases} (t, L_1 \cdots L_{n-2} t - \frac{1}{2}) & \text{if } 0 \le t \le \frac{1}{L_1 \cdots L_{n-2}} \\ \left(-t + \frac{2}{L_1 \cdots L_{n-2}}, \frac{L_1 \cdots L_{n-2} q_{n-1}}{q_{n-1} - L_1 \cdots L_{n-1}}(t - \frac{L_{n-1}}{q_{n-1}}) - \frac{1}{2}\right) & \text{if } \frac{1}{L_1 \cdots L_{n-2}} \le t \le \frac{L_{n-1}}{q_{n-1}} \end{cases} \\ \\ C_{n-1}\left(t + \frac{L_{n-1}}{q_{n-1}}\right) = R_{\frac{1}{q_{n-1}}} C_{n-1}(t). \end{cases}$$

$\ell_i = \left\{\left(t, L_1 \cdots L_{i-1} t - \frac{1}{2}\right) ; 0 \le t \le \frac{1}{L_1 \cdots L_{i-1}}\right\} (i = 2, 3, \cdots, n-1)$.
$\varphi_{n-2}|\ell_{n-1} = \psi_{n-2}|\ell_{n-1}$ (in particular, $\varphi_{n-2}(\ell_{n-1}) = \ell_{n-2}$).
$\theta_{n-1} = \sum_{i=1}^{n-1} \frac{1}{q_i}$.

We define $\psi_{n-1} : X_n \to X_{n-1}$ by $\psi_{n-1}(x, y) = C_{n-1}(L_{n-1} x)$. Then ψ_{n-1} maps X_n onto the curve C_{n-1}. Furthermore,

$$\begin{aligned} \psi_{n-1} R_{1/q_{n-1}}(x, y) &= C_{n-1}\left(L_{n-1} x + \frac{L_{n-1}}{q_{n-1}}\right) \\ &= R_{1/q_{n-1}} C_{n-1}(L_{n-1} x) \\ &= R_{1/q_{n-1}} \psi_{n-1}(x, y). \end{aligned}$$

Let ℓ_n denote the segment $\{(t, L_1 L_2 \cdots L_{n-1} t - \frac{1}{2}) ; 0 \le t \le \frac{1}{L_1 \cdots L_{n-1}}\}$. Then $\psi_{n-1}(\ell_n) = \ell_{n-1}$. We choose a C^∞-embedding $\varphi_{n-1} : X_n \to X_{n-1}$ along the curve C_{n-1} satisfying

(1) $d(\varphi_1 \cdots \varphi_{n-2} \varphi_{n-1}(x, y), \varphi_1 \cdots \varphi_{n-2} \psi_{n-1}(x, y)) < 1/2^{n+2}$ for any $(x, y) \in X_n$, In particular, $\mathrm{diam}\{\varphi_1 \cdots \varphi_{n-1}(x, y) ; |y| \le 1\} < 1/2^{n+1}$ because $\{\psi_{n-1}(x, y) ; |y| \le 1\}$ consists of one point.

(2) $\varphi_{n-1} \circ R_{1/q_{n-1}} = R_{1/q_{n-1}} \circ \varphi_{n-1}$. In particular, $\varphi_{n-1} \circ R_{\theta_{n-1}} = R_{\theta_{n-1}} \circ \varphi_{n-1}$.

(3) $\varphi_{n-1}|\ell_n = \psi_{n-1}|\ell_n$. In particular, $\varphi_{n-1}(\ell_n) = \ell_{n-1}$, $\varphi_{n-1}(0, -\frac{1}{2}) = (0, -\frac{1}{2})$ and $\varphi_{n-1}(\frac{1}{L_1 \cdots L_{n-1}}, \frac{1}{2}) = (\frac{1}{L_1 \cdots L_{n-2}}, \frac{1}{2})$.

(4) $\varphi_{n-1}(X_n) \subset \{(x, y) ; |y| < \frac{3}{4}\}$.

Let $\Phi_{n-1} = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1}$ for $n > 1$ ($\Phi_0 = \mathrm{id}$). We choose a large integer q_n satisfying that

(5) There is $k_n \in \mathbf{Z}_+$ such that $q_n = k_n q_{n-1} L_1 \cdots L_{n-1}$. In particular, $q_n > L_1 \cdots L_{n-1}$. Further we assume that $q_n > 2n$.

(6) If $z_1, z_2 \in X_n$ and $d(z_1, z_2) \le \frac{n}{q_n}$, then $d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_2)) < 1/2^n$.

(7) For $\theta_n = \sum_{i=1}^{n} \frac{1}{q_i}$, $q_n\theta_n$ and q_n are relatively prime. For example, if $q_n = k{q_{n-1}}^2$ and $\theta_n = \frac{1}{q_n} + \frac{j}{q_{n-1}}$ for some integers k and j, then $\theta_n = \frac{1+kjq_{n-1}}{k{q_{n-1}}^2}$. Thus $q_n\theta_n = 1 + kjq_{n-1}$ and $q_n = kq_{n-1}^2$ are relatively prime.

Here we remark that φ_{n-1} has already been given independent of the choice of q_n.

We choose a smooth increasing function $\eta_n : [\frac{3}{4}, 1] \to \mathbf{R}$ so that $\eta_n(3/4) = \theta_n$, $\eta_n(1) = \theta_{n-1}$ and η_n is constant on neighborhoods of $3/4$ and 1. We define a C^∞ diffeomorphism $f_n : X_1 \to X_1$ by

$$f_n(x,y) = \begin{cases} f_{n-1}(x,y) & \text{outside } \Phi_{n-1}(X_n) \\ \Phi_{n-1} R_{\eta_n(|t|)} \Phi_{n-1}^{-1}(x,y) & \text{if } (x,y) \in \Phi_{n-1}(X_n) \text{ and } \frac{3}{4} \le |p_2 \Phi_{n-1}^{-1}(x,y)| \le 1 \\ \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1}(x,y) & \text{if } (x,y) \in \Phi_{n-1}(X_n) \text{ and } |p_2 \Phi_{n-1}^{-1}(x,y)| \le \frac{3}{4}, \end{cases}$$

where f_n is well-defined by (2). We further assume that q_n is so large that f_n is assumed to be $1/2^n$-closed to f_{n-1} in the C^n-topology.

Let L_n denote the integer defined by $L_n = q_n \left(\frac{2}{L_1 \cdots L_{n-1}} - \frac{1}{q_n} \right)$. Then

$$\frac{1}{L_1 \cdots L_n} < \frac{1}{q_n} \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

because $\frac{1}{L_1 \cdots L_{n-1}} < \frac{2}{L_1 \cdots L_{n-1}} - \frac{1}{q_n} = \frac{L_n}{q_n}$. Thus we have

$$0 < \frac{1}{L_1 \cdots L_n} < \frac{1}{q_n} < \frac{1}{L_1 \cdots L_n} + \frac{1}{q_n} < \frac{2}{q_n} < \cdots < 1.$$

We define a simple closed curve $C_n : \mathbf{R}/L_n\mathbf{Z} \to X_n$ by

$$C_n(t) = \begin{cases} (t, L_1 \cdots L_{n-1} t - \frac{1}{2}) & \text{if } 0 \le t \le \frac{1}{L_1 \cdots L_{n-1}} \\ (-t + \frac{2}{L_1 \cdots L_{n-1}}, \frac{L_1 \cdots L_{n-1} q_n}{q_n - L_1 \cdots L_n}(t - \frac{L_n}{q_n}) - \frac{1}{2} & \text{if } \frac{1}{L_1 \cdots L_{n-1}} \le t \le \frac{L_n}{q_n} \end{cases}$$

and $C_n(t + \frac{L_n}{q_n}) = R_{\frac{1}{q_n}} C_n(t)$ for any t. Then C_n is invariant under R_{θ_n}. We construct φ_n and f_n ($n = 1, 2, \cdots$) inductively in this way.

By the same argument as in [6] and [4], we can choose q_n so large that f_n converges to a C^∞ diffeomorphism f as $n \to \infty$, and $d(f^k(x,y), f_n^k(x,y)) < 1/2^n$ for any $(x,y) \in X_1$ and $0 \le k \le q_n$.

Remark 1. We can extend f to a C^∞ diffeomorphism of any surface.

4.5 Properties of the minimal set.

Let $X = \bigcap_{n=2}^{\infty} \Phi_{n-1}(X_n)$. Then X is not empty because

$$\cdots \subset \Phi_n(X_{n+1}) \subset \Phi_{n-1}(X_n) \subset \cdots .$$

Furthermore, X contains the arc ℓ_1 because $\Phi_{n-1}(\ell_n) = \ell_1$. On the other hand, if $(x, y) \notin \Phi_k(X_{k+1})$ for some $k \in \mathbf{Z}_+$, then $f_n(x, y) = f_k(x, y)$ for any $n > k$. Since $\Phi_{n-1}(X_n)$ is connected, the set X is connected. Thus, in order to prove Theorem 2, we have only to show that X is a minimal set (Lemma 1) and X is not path-connected (Lemma 2).

Proposition 2. *For the subsets $D_i^n = \{(x, y) \in X_n \, ; \, \frac{i}{q_n} \leq x \leq \frac{i+1}{q_n}\}$ $(i = 0, 1, \cdots, q_n - 1)$, the diameter of $\Phi_{n-1}(D_i^n)$ is less than $\frac{1}{2^{n-1}}$.*

Proof. Let $z_1, z_2 \in D_i^n$. Let $z_1' = (p_1(z_1), 0)$ and $z_2' = (p_1(z_2), 0)$. Since $d(z_1', z_2') \leq \frac{1}{q_n}$, we have $d(\Phi_{n-1}(z_1'), \Phi_{n-1}(z_2')) < \frac{1}{2^n}$ by (6). Since $\{\psi_{n-1}(x, y) \, ; \, |y| \leq 1\}$ consists of one point, $\psi_{n-1}(z_i) = \psi_{n-1}(z_i')$ for $i = 1, 2$. Thus

$$\begin{aligned} d(\Phi_{n-1}(z_i), \Phi_{n-1}(z_i')) &\leq d(\Phi_{n-1}(z_i), \Phi_{n-2}\psi_{n-1}(z_i)) + d(\Phi_{n-2}\psi_{n-1}(z_i'), \Phi_{n-1}(z_i')) \\ &< \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} \\ &= \frac{1}{2^{n+1}} \end{aligned}$$

for $i = 1, 2$ by (1). Therefore

$$\begin{aligned} & d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_2)) \\ \leq\ & d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_1')) + d(\Phi_{n-1}(z_1'), \Phi_{n-1}(z_2')) + d(\Phi_{n-1}(z_2'), \Phi_{n-1}(z_2)) \\ <\ & \frac{1}{2^{n-1}}. \end{aligned}$$

■

Proposition 3. *For any z of X,*

$$f_n^j(z) = \Phi_{n-1} R_{\theta_n}^j \Phi_{n-1}^{-1}(z)$$

for $n, j \in \mathbf{Z}_+$.

Proof. Since $z \in \Phi_{n-1}\varphi_n(X_{n+1})$, we have $|p_2 \Phi_{n-1}^{-1}(z)| < \frac{3}{4}$ by (4). Therefore, $f_n(z) = \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1}(z)$. Furthermore, if $f_n^j(z) = \Phi_{n-1} R_{\theta_n}^j \Phi_{n-1}^{-1}(z)$ for some $j \in \mathbf{Z}_+$, then $|p_2 \Phi_{n-1}^{-1} f_n^j(z)| = |p_2 R_{\theta_n}^j \Phi_{n-1}^{-1}(z)| < \frac{3}{4}$ and $f_n^j(z) \in \Phi_{n-1}(X_n)$. Thus $f_n^{j+1}(z) = \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1} f_n^j(z) = \Phi_{n-1} R_{\theta_n}^{j+1} \Phi_{n-1}^{-1}(z)$. ■

Lemma 1. *X is a minimal set of f.*

Proof. First prove that X is invariant under f. Let $z \in X$. We fix $n \in \mathbf{Z}_+$. If $k \geq n$, then $f_k(z) = \Phi_{k-1} R_{\theta_k} \Phi_{k-1}^{-1}(z)$ by Proposition 3. Thus $f_k(z) = \Phi_{n-1}(\varphi_n \circ \cdots \circ \varphi_{k-1}) R_{\theta_k} \Phi_{k-1}^{-1}(z) \in \Phi_{n-1}(X_n)$. Therefore $f(z) = \lim_{k\to\infty} f_k(z) \in \Phi_{n-1}(X_n)$ for any n, and thus $f(z) \in X$. Since $f^{-1}(z) = \lim_{k\to\infty} f_k^{-1}(z)$, we can show that $f^{-1}(z) \in X$. Thus $f(X) = X$.

Let z and u be points of X. Let n be an arbitrary positive integer. Let $z_n = \Phi_{n-1}^{-1}(z) \in X_n$ and $u_n = \Phi_{n-1}^{-1}(u) \in X_n$. Then there is i $(0 \leq i < q_n)$ such that $u_n \in D_i^n = \{(x, y) \in X_n \,;\, \frac{i}{q_n} \leq x \leq \frac{i+1}{q_n}\}$. For $\theta_n = \frac{j_n}{q_n}$, the integers j_n and q_n are relatively prime by (7). Thus there is $k \in \mathbf{Z}$ $(0 \leq k < q_n)$ such that $R_{\theta_n}^k(z_n) \in D_i^n$. Since $\operatorname{diam} \Phi_{n-1}(D_i^n) < 1/2^{n-1}$ by Proposition 2, we have $d(\Phi_{n-1} R_{\theta_n}^k(z_n), \Phi_{n-1}(u_n)) < 1/2^{n-1}$.

On the other hand, by Proposition 3, $d(f_n^k(z), u) = d(\Phi_{n-1} R_{\theta_n}^k \Phi_{n-1}^{-1}(z), u) = d(\Phi_{n-1} R_{\theta_n}^k(z_n), \Phi_{n-1}(u_n)) < 1/2^{n-1}$ as above. Since $d(f^k(z), f_n^k(z)) < 1/2^n$ for $0 \leq k \leq q_n$ by construction, we conclude that $d(f^k(z), u) < \frac{3}{2^n}$. ■

We fix $n \geq 1$. Let $v_i = \left(\frac{i}{q_n}, -\frac{1}{2}\right) \in X_n$ and $w_i = \left(\frac{1}{L_1 \cdots L_{n-1}} + \frac{i}{q_n}, \frac{1}{2}\right) \in X_n$ for $i = 1, 2, \cdots, n$. Let $v_i' = \left(\frac{i}{q_n}, -\frac{1}{2}\right) \in X_{n+1}$ and $w_i' = \left(\frac{1}{L_1 \cdots L_n} + \frac{i}{q_n}, \frac{1}{2}\right) \in X_{n+1}$ for $i = 1, 2, \cdots, n$. Then $p_1(v_1') < p_1(w_1') < p_1(v_2') < p_1(w_2') < \cdots$ because $\frac{1}{L_1 \cdots L_n} < \frac{1}{q_n}$ (see (8)).

Proposition 4. *$\psi_n(v_i') = v_i$ and $\psi_n(w_i') = w_i$.*

Proof. $\psi_n(v_i') = \psi_n(\frac{i}{q_n}, -\frac{1}{2}) = C_n(i\frac{L_n}{q_n}) = (R_{1/q_n})^i C_n(0) = (\frac{i}{q_n}, -\frac{1}{2}) = v_i$. $\psi_n(w_i') = \psi_n(\frac{1}{L_1 \cdots L_n} + \frac{i}{q_n}, \frac{1}{2}) = C_n(\frac{1}{L_1 \cdots L_{n-1}} + \frac{iL_n}{q_n}) = (R_{1/q_n})^i C_n(\frac{1}{L_1 \cdots L_{n-1}}) = (\frac{1}{L_1 \cdots L_{n-1}} + \frac{i}{q_n}, \frac{1}{2}) = w_i$. ■

Lemma 2. *X is not path-connected.*

Proof. Let $z_1 = (1, \frac{1}{2}) \in X_1$ and $z_2 = (\frac{1}{2}, -\frac{1}{2}) \in X_1$. The point z_1 is an end point of ℓ_1. Thus $z_1 \in X$. Furthermore, $\Phi_n(\frac{1}{L_1 \cdots L_n}, \frac{1}{2}) = z_1$ for any $n \in \mathbf{Z}_+$ because $\Phi_n(\ell_{n+1}) = \ell_1$ by (3). On the other hand, for any $j \geq 1$, $\varphi_j(z_2) = \varphi_j(\frac{1}{q_1}, -\frac{1}{2}) = R_{\frac{1}{q_1}} \varphi_j(0, -\frac{1}{2}) = R_{\frac{1}{q_1}}(0, -\frac{1}{2}) = (\frac{1}{q_1}, -\frac{1}{2}) = z_2$ by (2). Thus $\Phi_n(\frac{1}{2}, -\frac{1}{2}) = z_2$ for any $n \in \mathbf{Z}_+$ and $z_2 \in X$.

Assume that there is a path γ connecting z_1 and z_2 contained in X. We further assume that $\gamma : [0, 1] \to X$ is homotopic to $t \mapsto (1 - \frac{1}{2}t, \frac{1}{2} - t)$ in X_1 with the boundary fixed (we can prove similarly in the other cases).

Let N denote the number of connected components of $\gamma \cap p_2^{-1}(-\frac{1}{4}, \frac{1}{4})$ such that one of the boundary points is contained in $p_2^{-1}(-\frac{1}{4})$ and the other boundary point is contained in $p_2^{-1}(\frac{1}{4})$.

We choose an integer n satisfying $2n-2>N$ and $n\geq 3$. Let $\gamma_{n+1} = \Phi_n^{-1}(\gamma)$. Then γ_{n+1} connects $(\frac{1}{L_1\cdots L_n},\frac{1}{2})$ with $(\frac{1}{q_1},-\frac{1}{2})$ in X_{n+1} as above. By (8) and (5), we obtain

$$\frac{1}{L_1\cdots L_n}<\frac{1}{q_n}<\frac{1}{q_n}+\frac{1}{L_1\cdots L_n}<\frac{2}{q_n}<\cdots<\frac{n-1}{q_n}+\frac{1}{L_1\cdots L_n}<\frac{n}{q_n}<\frac{1}{q_1}.$$

We choose points $a_i'\in X_{n+1}$ in $p_1^{-1}(\frac{i}{q_n})\cap\gamma_{n+1}$ for $i=1,2,\cdots,n$ and points $b_j'\in X_{n+1}$ in $p_1^{-1}(\frac{j}{q_n}+\frac{1}{L_1\cdots L_n})\cap\gamma_{n+1}$ for $j=1,2,\cdots,n-1$ so that there are s_i and t_j of $[0,1]$ satisfying $a_i'=\gamma_{n+1}(s_i)$, $b_j'=\gamma_{n+1}(t_j)$ and

$$0<s_1<t_1<s_2<t_2<\cdots<t_{n-1}<s_n<1.$$

For $i=1,2,\cdots,n$, $d(\Phi_n(v_i'),\Phi_n(a_i'))<1/2^{n+2}$ by (1). Furthermore, $d(\Phi_n(v_i'),\Phi_{n-1}(v_i))=d(\Phi_{n-1}\varphi_n(v_i'),\Phi_{n-1}\psi_n(v_i'))<1/2^{n+3}$ again by (1) and Proposition 4. Furthermore, $d(\Phi_{n-1}(v_i),(0,-\frac{1}{2}))<1/2^n$ by (6). As a result,

$$p_2\Phi_n(a_i')<-\frac{1}{2}+\frac{1}{2^n}+\frac{1}{2^{n+3}}+\frac{1}{2^{n+2}}<-\frac{1}{4}$$

when $n\geq 3$.

On the other hand, for $j=1,2,\cdots,n-1$, $d(\Phi_n(w_j'),\Phi_n(b_j'))<1/2^{n+2}$ by (1) and $d(\Phi_n(w_j'),\Phi_{n-1}(w_j))=d(\Phi_{n-1}\varphi_n(w_j'),\Phi_{n-1}\psi_n(w_j'))<1/2^{n+3}$ by (1) and Proposition 4. Since $d(\Phi_{n-1}(w_j),(1,\frac{1}{2}))=d(\Phi_{n-1}(w_j),\Phi_{n-1}(\frac{1}{L_1\cdots L_{n-1}},\frac{1}{2}))<1/2^n$ by (6), we have

$$p_2\Phi_n(b_j')>\frac{1}{2}-\frac{1}{2^n}-\frac{1}{2^{n+3}}-\frac{1}{2^{n+2}}>\frac{1}{4}$$

when $n\geq 3$.

The points $\Phi_n(a_i')$ and $\Phi_n(b_j')$ of γ satisfy $p_2\Phi_n(a_i')<-\frac{1}{4}$ and $p_2\Phi_n(b_j')>\frac{1}{4}$. Therefore, there are at least $2n-2$ connected components of $\gamma\cap p_2^{-1}(-\frac{1}{4},\frac{1}{4})$ such that one of the boundaries is contained in $p_2^{-1}(-\frac{1}{4})$ and the other boundary point is contained in $p_2^{-1}(\frac{1}{4})$. However, this contradicts the assumption, $2n-2>N$. ∎

Remark 2. A locally connected complete metric space is path-connected (see [7] §50). Thus the minimal set of Theorem 2 is not locally connected.

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