Restriction of the irreducible representations of GL_n to the symmetric group $\mathfrak{S}_n$.

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1 Problem

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a dominant integral weight of GL_n and consider the finite dimensional irreducible representation ρ_λ with the highest weight λ. The permutation matrices in GL_n form a finite subgroup which is isomorphic to the symmetric group $\mathfrak{S}_n$. We identify $\mathfrak{S}_n$ with this subgroup. Then our problem can be stated as follows.

Problem 1.1 *Describe the decomposition of $\rho_\lambda|_{\mathfrak{S}_n}$ when restricted to the subgroup $\mathfrak{S}_n$.*

We will reduce this problem to the decomposition of plethysms in principle.

2 Main result

Let us review $GL_n \times GL_m$-duality on the symmetric algebra $S(\mathbb{C}^n \otimes \mathbb{C}^m)$. It is well-known that $S(\mathbb{C}^n \otimes \mathbb{C}^m)$ is multiplicity free as a representation of $GL_n \times GL_m$ and decomposes as follows (see, e.g. [Howe]):

$$S(\mathbb{C}^n \otimes \mathbb{C}^m) \simeq \sum_{\mathrm{length}(\lambda) \leq \min\{m,n\}} \rho_\lambda^{(n)} \otimes \rho_\lambda^{(m)},$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a weight (or a partition) and $\rho_\lambda^{(n)}$ is the irreducible representation of GL_n with highest weight λ. Put $V = \mathbb{C}^m$ and assume $m \geq n$. Then we can reformulate the left hand side of the above formula:

$$\begin{aligned} S(\mathbb{C}^n \otimes V) &\simeq \otimes^n S(V) \\ &= \otimes^n \left(\bigoplus_{k=0}^{\infty} S_k(V) \right) \\ &= \bigoplus_{\nu=(\nu_1,\cdots,\nu_n) \in \mathbb{Z}_{\geq 0}^n} S_{\nu_1}(V) \otimes \cdots \otimes S_{\nu_n}(V), \end{aligned}$$

where $S_k(V)$ denotes a homogeneous component of degree k in $S(V)$. For $\nu = (\nu_1, \cdots, \nu_n) \in (\mathbb{Z}_{\geq 0})^n$, put

$$\mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_n} \subset \mathfrak{S}_{|\nu|} \quad (|\nu| = \nu_1 + \cdots + \nu_n).$$

Then we have

$$S_{\nu_1}(V) \otimes \cdots \otimes S_{\nu_n}(V) = \left(\otimes^{|\nu|} V\right)^{\mathfrak{S}_\nu}. \tag{2.1}$$

By the classical Schur duality (cf. [Weyl]), we know the decomposition

$$\otimes^{|\nu|} V \simeq \sum_{|\lambda|=|\nu|,\mathrm{length}(\lambda)\leq n} \rho_\lambda^{(m)} \otimes \sigma_\lambda$$

as a $GL_m \times \mathfrak{S}_{|\nu|}$-module. Therefore the above formula (2.1) becomes

$$\left(\sum_{|\lambda|=|\nu|,\mathrm{length}(\lambda)\leq n} \rho_\lambda^{(m)} \otimes \sigma_\lambda \right)^{\mathfrak{S}_\nu} = \sum_{|\lambda|=|\nu|,\mathrm{length}(\lambda)\leq n} \rho_\lambda^{(m)} \otimes (\sigma_\lambda)^{\mathfrak{S}_\nu}.$$

We summarize as

$$\begin{aligned} \otimes^m S(V) &\simeq \sum_{\nu \in \mathbb{Z}^n_{\geq 0}} \sum_{|\lambda|=|\nu|,\mathrm{length}(\lambda)\leq n} \rho_\lambda^{(m)} \otimes (\sigma_\lambda)^{\mathfrak{S}_\nu} \\ &\simeq \sum_{\mathrm{length}(\lambda)\leq n} \rho_\lambda^{(m)} \otimes \left(\sum_{\nu \in \mathbb{Z}^n_{\geq 0}, |\nu|=|\lambda|} (\sigma_\lambda)^{\mathfrak{S}_\nu} \right). \end{aligned}$$

Here, $\sum (\sigma_\lambda)^{\mathfrak{S}_\nu}$ becomes an $\mathfrak{S}_n$-module, whose module structure is induced by the original action of GL_n. So we obtain

$$\begin{aligned} \rho_\lambda^{(n)}\Big|_{\mathfrak{S}_n} &\simeq \sum_{\nu \in \mathbb{Z}^n_{\geq 0}, |\nu|=|\lambda|} (\sigma_\lambda)^{\mathfrak{S}_\nu} \\ &\simeq \sum_{\mu \vdash |\lambda|,\mathrm{length}(\mu)\leq n} \left(\sum_{\nu \in \mathfrak{S}_n \cdot \mu} (\sigma_\lambda)^{\mathfrak{S}_\nu} \right). \end{aligned}$$

Let $\mathcal{V}_\lambda$ be a representation space on which GL_n acts via $\rho_\lambda^{(n)}$ and $\mathcal{V}_\lambda(\mu)$ its weight space of weight μ. We put $\mathcal{V}_\lambda(\mathfrak{S}_n \cdot \mu) = \sum_{\nu \in \mathfrak{S}_n \cdot \mu} \mathcal{V}_\lambda(\nu)$. Then, clearly $\mathcal{V}_\lambda(\mathfrak{S}_n \cdot \mu)$ is invariant under $\mathfrak{S}_n$ and we get the following lemma.

Lemma 2.1 *As a representation of $\mathfrak{S}_n$, there is an isomorphism:*

$$\mathcal{V}_\lambda(\mathfrak{S}_n \cdot \mu) \simeq \sum_{\nu \in \mathfrak{S}_n \cdot \mu} (\sigma_\lambda)^{\mathfrak{S}_\nu}.$$

Take a partition $\mu \vdash |\lambda|$. Consider the normalizer $\mathfrak{N}_\mu = N_{\mathfrak{S}_{|\mu|}}(\mathfrak{S}_\mu)$ of $\mathfrak{S}_\mu$ in $\mathfrak{S}_{|\mu|}$;

$$\mathfrak{N}_\mu = \{s \in \mathfrak{S}_{|\mu|} \mid s\mathfrak{S}_\mu s^{-1} = \mathfrak{S}_\mu\}.$$

Then there exists a partition $\alpha(\mu) = \alpha = (\alpha_1, \cdots, \alpha_k)$ of $\mathrm{length}(\mu) \leq n$ such that

$$\mathfrak{N}_\mu/\mathfrak{S}_\mu \simeq \mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}.$$

Since $\mathfrak{N}_\mu$ normalizes $\mathfrak{S}_\mu$, it acts on $(\sigma_\lambda)^{\mathfrak{S}_\mu}$. Moreover, by definition, $\mathfrak{S}_\mu$ acts on $(\sigma_\lambda)^{\mathfrak{S}_\mu}$ trivially. So we get a representation of $\mathfrak{S}_\alpha \simeq \mathfrak{N}_\mu/\mathfrak{S}_\mu$ on $(\sigma_\lambda)^{\mathfrak{S}_\mu}$.

Proposition 2.2 *With the notations above, we have*

$$\sum_{\mu \in \mathfrak{S}_n \cdot \mu} (\sigma_\lambda)^{\mathfrak{S}_\mu} \simeq \mathrm{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_{n-|\alpha|}}^{\mathfrak{S}_n} (\sigma_\lambda)^{\mathfrak{S}_\mu} \otimes 1,$$

where 1 *means the trivial representation of* $\mathfrak{S}_{n-|\alpha|}$.

Proof. ∎

Now we summarize the above results into

Theorem 2.3

$$\left.\rho_\lambda^{(n)}\right|_{\mathfrak{S}_n} \simeq \bigoplus_{\mu \vdash |\lambda|} \mathrm{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_{n-|\alpha|}}^{\mathfrak{S}_n} (\sigma_\lambda)^{\mathfrak{S}_\mu} \otimes 1,$$

where $\alpha \vdash \mathrm{length}(\mu)$ *is determined by* μ *via*

$$N_{\mathfrak{S}_{|\lambda|}}(\mathfrak{S}_\mu)/\mathfrak{S}_\mu \simeq \mathfrak{S}_\alpha.$$

Remark. Put $k = |\lambda| = |\mu|$. By the Frobenius reciprocity, we have

$$(\sigma_\lambda)^{\mathfrak{S}_\mu} \simeq \mathrm{Hom}_{\mathfrak{S}_\mu}(\mathbb{C}, \sigma_\lambda) \simeq \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_k}\mathbb{C}, \sigma_\lambda),$$

$$\Longrightarrow \quad \dim(\sigma_\lambda)^{\mathfrak{S}_\mu} = [\sigma_\lambda : \mathrm{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_k}\mathbb{C}].$$

This number is called the Kostka number (cf. [Macdonald]). By Lemma 2.1, it is equal to the weight multiplicity of μ in the representation $(\rho_\lambda^{(n)}, \mathcal{V}_\lambda)$ of GL_n.

3 Relation to the plethysm

It seems difficult to determine the action of $\mathfrak{S}_\alpha$ on $(\sigma_\lambda)^{\mathfrak{S}_\mu}$. The reason why it is difficult is as follows. Put $k = |\lambda|$ (i.e., $\lambda \vdash k$). Note that the normalizer of $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_n}$ in $\mathfrak{S}_k$ is a wreath product

$$\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu \subset \mathfrak{S}_k, \; \mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_m}$$

for some $\alpha \vdash n$. The irreducible representations of a wreath product are well-studied and completely classified (see [JK, Theorem 4.3.34]).

In our case, for $\pi \in \mathfrak{S}_\alpha^\wedge$, let $(\pi : 1)$ be an irreducible representation of $\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu$ which is obtained by the successive application of the natural projection of $\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu \to \mathfrak{S}_\alpha$ and the representation π of $\mathfrak{S}_\alpha$. Then we get

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{S}_\alpha}(\pi, (\sigma_\lambda)^{\mathfrak{S}_\mu}) &= \mathrm{Hom}_{\mathfrak{S}_\alpha}((\pi : 1)^{\mathfrak{S}_\mu}, (\sigma_\lambda)^{\mathfrak{S}_\mu}) \\ &= \mathrm{Hom}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}((\pi : 1), \sigma_\lambda) \\ &\simeq \mathrm{Hom}_{\mathfrak{S}_k}(\mathrm{Ind}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}^{\mathfrak{S}_k}(\pi : 1), \sigma_\lambda). \end{aligned}$$

If we know $(\sigma_\lambda)^{\mathfrak{S}_\mu}$ completely as an $\mathfrak{S}_\alpha$-module, then we know $\mathrm{Hom}_{\mathfrak{S}_\alpha}(\pi, (\sigma_\lambda)^{\mathfrak{S}_\mu})$, hence the multiplicity of σ_λ in $\mathrm{Ind}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}^{\mathfrak{S}_k}(\pi : 1)$ can be determined. However, the induced representation

$$\mathrm{Ind}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}^{\mathfrak{S}_k}(\pi : 1)$$

is a special case of plethysms and its decomposition is not completely known yet (cf. [JK, §5.4]).

Using the above relation, we can rephrase our theorem in terms of the decomposition of plethysm:

$$(\sigma_\lambda)^{\mathfrak{S}_\mu} \simeq \bigoplus_{\pi \in \mathfrak{S}_\alpha^\wedge} [\sigma_\lambda : \mathrm{Ind}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}^{\mathfrak{S}_k}(\pi : 1)] \; \pi.$$

Theorem 3.1

$$\rho_\lambda^{(n)}\Big|_{\mathfrak{S}_n} \simeq \bigoplus_{\mu \vdash |\lambda|} \bigoplus_{\pi \in \mathfrak{S}_\alpha^\wedge} [\sigma_\lambda : \mathrm{Ind}_{\mathfrak{S}_\alpha \ltimes \mathfrak{S}_\mu}^{\mathfrak{S}_k}(\pi : 1)] \; \mathrm{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_{n-|\alpha|}}^{\mathfrak{S}_n} \pi \otimes 1.$$

Example 3.2 Take $\lambda = (k^n) = (k, \cdots, k)$. In this case, we have $\rho_\lambda^{(n)} = (\det{}^{(n)})^k$. So, we know $\rho_\lambda^{(n)}\Big|_{\mathfrak{S}_n} = (\mathrm{sgn})^k$. Moreover, by Lemma 2.1, we get

$$(\sigma_\lambda)^{\mathfrak{S}_\mu} = \begin{cases} \mathbb{C} & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu = \lambda$, the normalizer of $\mathfrak{S}_\lambda = \mathfrak{S}_k \times \cdots \times \mathfrak{S}_k$ in $\mathfrak{S}_{nk}$ is isomorphic to $\mathfrak{S}_n \ltimes (\mathfrak{S}_k)^n$. So we have

$$[\sigma_{(k^n)}, \mathrm{Ind}_{\mathfrak{S}_n \ltimes (\mathfrak{S}_k)^n}^{\mathfrak{S}_{nk}}(\pi : 1)] = \begin{cases} 1 & \text{if } \pi = (\mathrm{sgn})^k, \\ 0 & \text{if } \pi \neq (\mathrm{sgn})^k. \end{cases}$$

Example 3.3 $|\lambda| = n$ and $\mu = (1, \cdots, 1) = (1^n)$. Then $\alpha(\mu) = (n)$ and $\mathcal{V}(\mu) \simeq \sigma_\lambda$. This is a well-known result (cf. [Kostant]).

Example 3.4 $|\lambda| = kn$ and $\mu = (k, \cdots, k) = (k^n)$. Then $\alpha(\mu) = (n)$ and

$$\mathcal{V}(\mu) \simeq \bigoplus_{\pi \in \widehat{\mathfrak{S}}_n} \left[\sigma_\lambda : \mathrm{Ind}_{\mathfrak{S}_n \ltimes (\mathfrak{S}_k)^n}^{\mathfrak{S}_{nk}} (\pi : 1) \right] \pi$$

This is a result of [AMT].

References

[AMT] S. Ariki, J. Matsuzawa and I. Terada, Representations of Weyl groups on zero weight spaces. In "Algebraic and Topological Theories, to the memory of Dr. Takehiko Miyata", Kinokuniya, 1985, pp. 546 – 568.

[Gutkin] E. Gutkin, Representations of the Weyl group in the space of vectors of zero weight. *UspehiMat. Nauka*, **28**(1973), 237 – 238 (in Russian).

[Kostant] B. Kostant, On Macdonald's η-function formula, the Laplacian and generalized exponents. *Adv. in Math.*, **20**(????), 257 – 285.

[King] R. C. King, Branching rules for $GL(N) \supset \Sigma_m$ and the evaluation of inner plethysms, *J. Math. Phys.*, **15**(1974).

[Howe] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In "The Schur lectures (1992)", Israel Mathematical Conference Proceedings, **8**(1995), Bar-Ilan Univ., pp. 1 – 182.

[JK] G. James and A. Kerber, *The representation theory of the symmetric group.* Encyclopedia of Math. and App. **16**, Addison-Wesley, 1981.

[Macdonald] I. Macdonald, *Symmetric functions and Hall polynomials.* Clarendon Press, 1979.

[Weyl] H. Weyl, *The Classical Groups.* Princeton University Press, Princeton, N.J., 1946.

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