Cells in Weyl group

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1 Introduction to cells

$$\text{cells:} \begin{cases} *\text{primitive ideals of } U(\mathfrak{g}) \ (\mathfrak{g}: \text{ semisimple Lie algebra}) \\ *\text{representation theory of finite Chevalley groups (unipotent representations)} \\ *\text{modular representation theory} \\ \ \text{etc.} \end{cases}$$

Since I am not a specialist of finite Chevalley groups or modular representation theory, I will explain properties of cells through *primitive ideals of* $U(\mathfrak{g})$.

Role in the theory of primitive ideals:

$\mathfrak{g}$: a semisimple Lie algebra $/\mathbb{C}$
$U(\mathfrak{g})$: the enveloping algebra $\supset \mathfrak{Z}$: center
$\mathfrak{g} \supset \mathfrak{h}$: CSA, $W = W(\mathfrak{g}, \mathfrak{h})$: Weyl group

$$\mathfrak{Z} \stackrel{\sim}{\to} U(\mathfrak{h})^W \ : \text{Harish-Chandra isomorphism}$$

For $\lambda \in \mathfrak{h}^*$, $\exists \chi_\lambda$:

$$\begin{array}{ccccc} \mathfrak{Z} & & \longrightarrow & & \mathbb{C} \ \text{: algebra hom.} \\ & \searrow & & \nearrow & \text{evaluation at } \lambda \\ & & U(\mathfrak{h})^W = S(\mathfrak{h})^W & & \end{array}$$

χ_λ : the central character corresponding to λ: Note that $\chi_\lambda = \chi_{w\lambda} \quad (w \in W)$.

L : an irreducible left $U(\mathfrak{g})$-module

$$\operatorname{Ann} L = \{X \in U(\mathfrak{g}) \mid Xl = 0 \ (\forall l \in L)\} : \text{ primitive ideal}^1$$

$$\operatorname{Ann} L \cap \mathfrak{Z} = \ker \chi_\lambda \ (\exists \lambda \in \mathfrak{h}^*) \stackrel{\text{def.}}{\Longleftrightarrow} \operatorname{Ann} L \in \operatorname{Prim}(\lambda) : \text{ primitive ideals with central character } \chi_\lambda$$

$$\operatorname{Prim} U(\mathfrak{g}) = \coprod_{\lambda \in \mathfrak{h}^*/W} \operatorname{Prim}(\lambda)$$

Example 1.1 $\operatorname{Prim}(\rho) \ni \operatorname{Ann}(\text{trivial rep.})$

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[1] There are several notions of primitivity in $U(\mathfrak{g})$ (cf. Dixmier [Dixmier]).

$$\begin{array}{ccccc} I : \text{maximal} & & I : \text{completely prime} & & \\ \Downarrow & & \Downarrow & & \\ I : \text{primitive} & \Rightarrow & I : \text{prime} & \Rightarrow & I : \text{semi-prime} \end{array}$$

$A \ni 1$: ring
(1) $I \subset A$: prime ideal $\Leftrightarrow \left(\begin{array}{l} A/I \supset \forall J_1, J_2 : \text{ideals } \neq 0 \\ \Rightarrow \ J_1 \cdot J_2 \neq (0) \end{array} \right)$
(2) $I \subset A$: completely prime $\Leftrightarrow$ A/I : integral domain
(3) $I \subset A$: semi-prime ideal $\Leftrightarrow \left(\begin{array}{l} A/I \supset J : \text{nilpotent ideal} \\ \Rightarrow \ J = (0) \end{array} \right)$

By translation principle, $\mathrm{Prim}(\rho) \simeq \mathrm{Prim}(\rho+\lambda)\ \forall\lambda \in P^+(\Delta)$
$\Delta = \Delta(\mathfrak{g},\mathfrak{h})$: roots
$P^+(\Delta)$: dominant integral weights

Parametrization of $\mathrm{Prim}(\rho)$**?**
$w \in W$

$$\begin{array}{rll} M_w = & M(w\rho-\rho) & : \text{Verma module with h.w. } w\rho-\rho \\ & \downarrow\ : \text{surjection} & \\ L_w = & L(w\rho-\rho) & : \text{its irreducible quotient} \end{array}$$

Theorem 1.2 (Duflo, 1977 [Duflo]) $\varphi : W \ni w \mapsto \mathrm{Ann}\, L_w \in \mathrm{Prim}(\rho)$ *is surjective.*

Actually Duflo proved more. He proved the theorem for general central character χ_λ's, which are possibly singular.

Classification of $\mathrm{Prim}(\rho) \Leftrightarrow$ determination of fibers of φ

$$\Leftrightarrow \begin{cases} \text{description of equivalence relation on } W : \\ w \sim w' \overset{\text{def.}}{\Longleftrightarrow} \varphi(w) = \varphi(w') \end{cases}$$

This equivalence relation defines the left cells in W, i.e., $\Gamma = \Gamma(w) = \{w' |\ \varphi(w) = \varphi(w')\}$ is a left cell (Joseph and Vogan, 1980).[2]
Write $\underset{\mathrm{L}}{\sim}$ instead of $\sim$.

Example 1.3 type A_{n-1}. $W \simeq \mathfrak{S}_n$: symmetric group of order n

$$\mathfrak{S}_n \ni w \underset{\text{Robinson-Schensted}}{\longmapsto} (T_w^L, T_w^R) \quad : \text{pair of standard tableaux of the same shape}$$

$w \underset{\mathrm{L}}{\sim} w' \Leftrightarrow T_w^L = T_{w'}^L$ (Barbasch-Vogan, 1982 [BVI, p.171]).[3]

$\mathfrak{h}^* \ni \mu \mapsto \mathrm{rk}(U(\mathfrak{g})/\,\mathrm{Ann}\, L(\mu-\rho)) =: p(\mu)$ rk : Goldie rank[4].
$P'(\Delta)$: regular integral weights, $P(\Delta)^{++} = \{\mu \in P'(\Delta) |\ \mu : \text{dominant}\}$

Theorem 1.4 (Joseph, 1980 [JI, Cor. 5.12]) *For $w \in W$, $p(w\mu)$ is a polynomial on $P(\Delta)^{++}$. Denote this polynomial by $p_w(\mu)$: Joseph's Goldie rank polynomial.*

Remark. Using the notion of "coherent family", this theorem is more comprehensive. $p_w(\mu)$ coincides with the character polynomial (up to scalar multiple) of the coherent family containing L_w. (Joseph, 1980 [JII, Theorem 5.1]. Cf. King, 1981 [King].)

[2] This is the definition of left cell Γ here, but historically, the left cells are defined by Lusztig by using purely algebraic structures of Coxeter groups. The equivalence of two definitions are established by Joseph and Vogan. See Theorem 1.6.

[3] If $\underset{\mathrm{R}}{\sim}$ is similarly defined by using the right annihilators, then $w \underset{\mathrm{R}}{\sim} w' \Leftrightarrow T_w^R = T_{w'}^R$.
For two-sided cells, $w \underset{\mathrm{LR}}{\sim} w' \Leftrightarrow T_w^L$ and $T_{w'}^L$ have the same shape $\Leftrightarrow T_w^R$ and $T_{w'}^R$ have the same shape .

[4] A : a prime ring (i.e., (0) is a primitive ideal $\Leftrightarrow \exists$ faithful irreducible representation)
Then $\mathrm{Fract}(A) \simeq Mat(n \times n, D)$ for some division ring D. Define $\mathrm{rk}\, A := n$.

$p_w(\mu)$: a harmonic polynomial on $\mathfrak{h}^*$, generates an irreducible W-module.

Γ : left cell $\ni w, w' \Rightarrow p_w(\mu) = p_{w'}(\mu)$ (by definition they take the same value on $\rho +$ they are proportional. See [JI, Cor.5.12])
$\Longrightarrow$ Write $p_\Gamma(\mu) = p_w(\mu) \quad (w \in \Gamma)$.

$\forall \sigma \in W^\vee$

$$\mathcal{C}_\sigma := \coprod \{\Gamma | \mathbb{C}W p_\Gamma \simeq \sigma\} \subset W$$

$\mathcal{C} = \mathcal{C}_\sigma$: *two-sided cell* in W (if $\mathcal{C}_\sigma \neq \emptyset$[5])
equivalence relation $w \underset{\mathrm{LR}}{\sim} w'$ ($\Leftrightarrow w, w' \in \exists \mathcal{C}$)

$\sigma \in W^\vee$: *special representation*[6] $\overset{\text{def.}}{\Leftrightarrow}$ $\mathcal{C}_\sigma \neq \emptyset$

$$\dim \sigma = \#\{\Gamma | \Gamma \subset \mathcal{C}_\sigma : \text{ left cell } \} \Rightarrow \#\mathrm{Prim}(\rho) = \sum_{\sigma \in W^\vee \,:\, \text{special}} \dim \sigma$$

Example 1.5 type A_{n-1}. $\forall \sigma \in \mathfrak{S}_n^\vee$: special $\leftrightarrow Y_\sigma$: Young diagram

$$\begin{array}{ccl} \mathcal{C}_\sigma & = & \{w \leftrightarrow (T_w^L, T_w^R) | \; T_w^L \text{ has the shape } Y_\sigma\} \\ \cup & & \\ \Gamma & = & \{(T_w^L, T_w^R)\} : \text{ for the fixed standard tableau } T_w^L \end{array}$$

$$\Longrightarrow \quad \#\{\Gamma | \; \Gamma \subset \mathcal{C}_\sigma\} = \#\{ \text{ standard tableau of the shape } Y_\sigma\} = \dim \sigma$$

$$\begin{array}{rcl} \#\mathrm{Prim}(\rho) & = & \displaystyle\sum_{\sigma \in \mathfrak{S}_n^\vee} \dim \sigma = \#\{ \text{ involutions in } \mathfrak{S}_n\} \\ \#\{\Gamma \subset \mathcal{C}\} & = & \#\{ \text{ involutions } \in \mathcal{C}\} \quad \text{([Duflo, Proposition 9], cf. [Borho, §5.9])} \end{array}$$

Remark. The above left (or two-sided) cells can be defined by using only combinatoric properties of a Coxeter system (by Kazhdan-Lusztig [KL]). So *they are defined for general Coxeter groups*, not only for Weyl groups. See [BVII, Remark after Cor.2.16].

$$\mathrm{Char}\, L_w = \sum_{w' \in W} a_{w,w'} \,\mathrm{Char}\, M_{w'} \; (a_{w,w'} \in \mathbb{Z})$$

$a_{w,w'}$ is explicitly determined (Kazhdan-Lusztig conjecture). But we do not need the explicit information.

$$\begin{array}{rcl} a(w) & := & \displaystyle\sum_{w' \in W} a_{w,w'} w' \in \mathbb{C}W \\ D(w) & := & \{w' | \; a(w') \in [\mathbb{C}W a(w)]\} = \{w' | \; \mathrm{Ann}\, L_{w'} \supset \mathrm{Ann}\, L_w\} \end{array}$$

The second equality is due to Vogan [Vogan].

[5] This is not empty iff σ is a special representation. See below.

[6] Special representations are originally introduced by Lusztig ([LI, LII]). However, two different notions coincide ([BVII, Theorem 2.29]).

Here, $[S]$ means **a**-basal subspace[7] in $\mathbb{C}W$ spanned by S.

Theorem 1.6 (Joseph [Joseph], Vogan [Vogan])

$$w' \underset{\mathrm{L}}{\sim} w \iff [\mathbb{C}W a(w')] = [\mathbb{C}W a(w)]$$

For Γ : a left cell ($w \in \Gamma$), we put[8]

$$\sigma_\Gamma := \sum_{w' \in D(w)} [\mathbb{C}W a(w')] \Big/ \sum_{w'' \in D(w), w'' \notin \Gamma} [\mathbb{C}W a(w'')] \quad : \text{left cell representation, not irreducible in general}$$

Similarly put

$$C(w) = \{w' | \ a(w') \in [\mathbb{C}W a(w) W]\}$$

Theorem 1.7 ([BVII, Cor.2.16])

$$w \underset{\mathrm{LR}}{\sim} w' \iff [\mathbb{C}W a(w') W] = [\mathbb{C}W a(w) W]$$

For $\mathcal{C}$: a two-sided cell ($w \in \mathcal{C}$), we put[9]

$$\sigma_{\mathcal{C}} := \sum_{w' \in C(w)} [\mathbb{C}W a(w') W] \Big/ \sum_{w'' \in C(w), w'' \notin \mathcal{C}} [\mathbb{C}W a(w'') W] \quad : \text{two-sided cell representation}$$

As a $W \times W$-module,

$$\sigma_{\mathcal{C}} = \sum_{\tau}^{\oplus} \tau \otimes \tau \text{ (multiplicity free)}$$

$$\mathcal{F}_{\mathcal{C}} := \{\tau \in W^\vee | \ [\sigma_{\mathcal{C}} : \tau \otimes \tau] = 1\} \subset W^\vee : \text{a } \textit{family} \text{ associated to } \mathcal{C}$$

$\mathcal{F} = \mathcal{F}_{\mathcal{C}}$ is also called two-sided cell[10] in $W^\vee$.

$$W^\vee = \coprod_{\mathcal{C} : \text{ two-sided cell}} \mathcal{F}_{\mathcal{C}}$$

$\forall \mathcal{F}_{\mathcal{C}}$ contains exactly one special representation, i.e., {special representations $\in W^\vee$} gives a complete system of representatives for families.

[7] A subspace in $\mathbb{C}W$ is called **a**-basal if it has basis consisting of $a(w)$'s. See Joseph II 298p. Note that $\{a(w) | \ w \in W\}$ forms a basis of $\mathbb{C}W$. In the original version of this note, I misunderstood the definition of $D(w)$; Tanisaki kindly pointed it out.

[8] The first summation in the right hand side of the next formula is surplus.

[9] Here, the summation is also surplus.

[10] Original definition of family is very complicated one ([Orange Book, §4.2]). However it coincides with the definition presented here (Barbasch-Vogan [BVII, Theorem 2.29], cf. [Orange Book, Theorem 5.25]).

2 Invariant for cells

We want to define an invariant for families $\{\mathcal{F}\}$.
The present observations are experimental ones[11].

Gyoja(1983) [Gyoja] : $\sigma \in H_q(W)^\vee$, $H_q(W)$: an Iwahori-Hecke algebra[12]

$$L(t,\sigma) = L(t,q,\sigma) := \sum_{w\in W} \sigma(w) t^{l(w)} \ (l(w) : \text{length function})$$

$L(t,\sigma)$: $\begin{cases} \exists \text{ functional equation} \\ \text{distribution of zeros and poles}^{13} \end{cases}$ are similar to zeta functions[14](congruence zeta functions on algebraic varieties).

$\overset{\text{Iwahori's suggestion}}{\Longrightarrow}$ Take $l'(w)$ instead of $l(w)$;

$$l'(w) := \begin{pmatrix} \text{minimal of } m, \text{ where } w = r_1 r_2 \cdots r_m, \\ r_i : \text{reflection (not necessarily simple)} \end{pmatrix}$$

(Cf. for type A, $\#\mathrm{Prim}(\rho) = \#\{\text{involutions}\}$)

Remark. $W \curvearrowright V$: natural representation, $V(w) := \{v \in V \mid wv = v\}$ $(w \in W)$.
Then $l'(w) = \operatorname{codim} V(w)$.

Gyoja-N.-Shimura :
$\sigma \in W^\vee$, W : *not* an Iwahori-Hecke algebra, but a *finite* Weyl group.

$$c(\sigma, t) := \sum_{w\in W} (\operatorname{trace} \sigma(w)) t^{l'(w)}$$

Does $c(\sigma,t)$ classify cells? : ($c(\sigma,t)$ is computable!)
For types $A_l, B_l(=C_l), G_2$: yes! i.e.,

$$\sigma_1 \underset{\mathrm{LR}}{\sim} \sigma_2^\iota \ (\iota \in \mathrm{Aut}(W,S)) \ \Leftrightarrow \ c(\sigma_1,t) = c(\sigma_2,t)$$

(σ_1 and σ_2^ι belong to the same family)

For Weyl groups of the other types, some deviation occurs.

[11] M. Sato: Every mathematician should be an experimentalist.

[12] For a long time, $H_q(W)$ is simply called *Hecke algebra*. Recently, it becomes to be called *Iwahori-*Hecke algebra. For this, see Introduction of [Orange Book].

[14] In case of affine Weyl groups, poles appear. However, in this note, Weyl groups always remain finite.

[14] Functional equation:

$$L(t,\sigma) = \sigma(T_{w_0}) t^{l(w_0)} L((-qt)^{-1}, \widehat{\sigma}), \ w_0 : \text{the longest element}, \quad \widehat{\sigma}(T_w) = (-q)^{l(w)} \left(T_{w^{-1}}\right)^{-1}.$$

zeros:

$$t = \zeta q^{-i/2m}, \ \zeta : \text{root of unity}, \quad i, m \in \mathbb{Z} \text{ s.t. } 1 \le m \le l(w_0), 0 \le i/2m \le 1.$$

Want to avoid the deviation.

$W \curvearrowright V$: natural representation, $\{m_i\}$: exponents, $H(V) = \oplus_n H^n(V)$: harmonic polynomials

$\sigma \in W^\vee, \chi = \mathrm{Char}\,\sigma$

$$\begin{aligned}
\tilde{\tau}(q,y)(w) &:= \frac{\det(1+yw|V)}{\det(1-qw|V)}, \\
\tilde{\tau}(\sigma;q,y) &:= \langle \chi, \tilde{\tau}(q,y)\rangle_W \\
&= \frac{1}{\#W}\sum_{w\in W}\chi(w^{-1})\tilde{\tau}(q,y)(w) \\
&= \sum_{i,j}\langle S^i(V)\otimes \wedge^j(V), \chi\rangle_W q^i y^j \\
&= \frac{\dim\sigma}{\prod_{i=1}^{l}(1-q^{m_i+1})}\sum_{n,m\geq 0}\left[\chi : H^n(V)\otimes \bigwedge{}^m(V)\right] q^n y^m,
\end{aligned}$$

Proposition 2.1 $l := \dim V$: *rank of* W

$$\frac{\#W}{\dim\sigma}\lim_{q\to 1}\tilde{\tau}(\sigma;q,-1+t(1-q)) = t^l c(\sigma;t^{-1})$$

$\implies$ $\tilde{\tau}(\sigma;q,y)$ refines the property of $c(\sigma,t)$.

From now on, W *is assumed to be irreducible.*

Observation 2.2 $\tilde{\tau}(\sigma;q,y)$ *has many factors of the same type* $(1+q^c y)$ $(c\in\mathbb{Z})$.

Example 2.3 (Or a proposition)

$$\#(\text{family}) = 1 \;\Rightarrow \tilde{\tau}(\sigma;q,y) = f(q)\prod_{i=1}^{\text{rank}}(1+q^{c_i}y)$$

$$\tilde{\tau}(\text{trivial};q,y) = \prod_{i=1}^{l}\frac{1+yq^{m_i}}{1-q^{m_i+1}}$$

Example 2.4 (See §3) type A_{l-1} : $W \simeq \mathfrak{S}_l$

$$\tilde{\tau}(\chi^\lambda;q,y) = q^{n(\lambda)}\prod_{x\in\lambda}\frac{1+q^{c(x)}y}{1-q^{h(x)}}\ \lambda : \text{partition and tableau}$$

notations: (cf. Macdonald's book [M])

$$n(\lambda) = \sum(i-1)\lambda_i,\ x=(i,j)\in\lambda\in\mathbb{Z}^2,\ c(x) = j-i,\ h(x) = \text{hook length}$$

λ :

☐	☐	☐	☐	☐
☐	☐	☐	☐	
☐	☐			
☐				

$c(x)$:

0	1	2	3	4
-1	0	1	2	
-2	-1			
-3				

, $h(x)$:

8	6	4	3	1
6	4	2	1	
3	1			
1				

We put $\tau_1(\sigma; q, y) = \prod_{i=1}^{\kappa(\sigma)} (1+q^{c_i}y)$: the largest factor of $\tilde{\tau}(\sigma; q, y)$ including only the factors $(1 + q^c y)$ $(c \in \mathbb{Z})$.

Theorem 2.5 *If*
(1) *W is of type $A_l, B_l (= C_l), G_2$ or*
(2) *W is of type D_l, F_4, E_l $(l = 6, 7, 8)$ and $\#(\text{family}) \leq 3$,*
then

$$\sigma_1 \underset{\text{LR}}{\sim} \sigma_2^\iota \ (\iota \in \text{Aut}(W, S)) \ \Leftrightarrow \ \tau_1(\sigma_1; q, y) = \tau_1(\sigma_2; q, y)$$

i.e., σ_1 and σ_2^ι belong to the same family.

Remark. Even for the case $\#(\text{family}) \geq 5$, the above theorem is *almost* true. We can attach linear factors like $\tau_1(\sigma; q, y)$ for each $\sigma \in W^\vee$, but it is not the largest linear factor (i.e., we must choose “special” linear factors).

Remark. If $\#(\text{family}) = 3$,

$$\tilde{\tau}(\sigma_1; q, y) + \tilde{\tau}(\sigma_2; q, y) = f(q) \prod_i (1 + q^{d_i} y) \quad (d_i \in \mathbb{Z})$$

for any two representations σ_1, σ_2 in the family.
Similar phenomena occur for the case of type D_l (and $\#(\text{family}) \geq 5$).

The set of (computable) integers

$$(c_1, c_2, \cdots, c_{\kappa(\sigma)})$$

determines a family (or two-sided cell) in $W^\vee$ (for W in the theorem).
If we overcome the deviation explained above (i.e., the case of $\#(\text{family}) \geq 5$), the computable invariants (c_i) really classifies families (or two-sided cells) in $W^\vee$ (even for non-irreducible W's).

Problem 2.6 (1) *Find a method to choose special linear factors for exceptions of the above theorem.*
(2) *The above consideration is an experimental one. Clarify the meaning of the invariants (c_i) from the view points of the representation theory of Weyl groups or Iwahori-Hecke algebras, theory of primitive ideals, finite Chevalley groups, modular representation theory* ....

3 Calculation of $\tilde{\tau}$ for type A

This section is almost borrowed from Macdonald's book: [M].

First, let us introduce some general theory from [M, §I.2 and §I.3]. In the following, t is an indeterminate and h_r, e_r, p_r are coefficients;

$$
\begin{aligned}
H(t) &= \sum_{r\geq 0} h_r t^r \; (h_0 = 1) \\
E(t) &:= 1/H(-t) = \sum_{r\geq 0} e_r t^r \\
P(t) &:= \frac{d}{dt}\log H(t) = H'(t)/H(t) = \sum_{r\geq 0} p_r t^r
\end{aligned}
$$

For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, put

$$h_\lambda = \prod_{i\geq 1} h_i^{\lambda_i}, \quad e_\lambda = \prod_{i\geq 1} e_i^{\lambda_i}, \quad p_\lambda = \prod_{i\geq 1} p_i^{\lambda_i}.$$

Theorem 3.1

$$h_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda} p_\lambda, \; e_r = \sum_{|\lambda|=r} \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda$$

Here, we used the following notation;

$$
\begin{aligned}
z_\lambda &:= \prod_{i\geq 1} i^{m_i} m_i! \; (\lambda = (1^{m_1} \cdot 2^{m_2} \cdot 3^{m_3} \cdots) \quad i.e., \; m_i = \#\{j \mid \lambda_j = i\}) \\
&= \frac{\#(\textit{conjugacy class in } \mathfrak{S}_n \textit{ corresponding to } \lambda)}{n!} \quad (n = |\lambda|) \\
\varepsilon_\lambda &:= (-1)^{|\lambda| - l(\lambda)}
\end{aligned}
$$

Theorem 3.2 *Put*

$$
\begin{aligned}
s_\lambda &:= \det(h_{\lambda_i - i + j})_{1\leq i,j\leq n} \; (n \geq l(\lambda)) \\
&= \sum_{w\in\mathfrak{S}_n} \varepsilon(w) h_{\lambda+\delta-w\delta} \\
&= \det(e_{\lambda'_i - i + j})_{1\leq i,j\leq m} \; (m \geq l(\lambda'))
\end{aligned}
$$

Now put $n := |\lambda| (\geq l(\lambda))$ and take the character χ^λ of $\mathfrak{S}_n$ corresponding to λ. Then we have

$$s_\lambda = \sum_\mu \frac{1}{z_\mu} \chi^\lambda(\mu) p_\mu,$$

where μ runs over all the partitions of n (i.e., conjugacy classes of $\mathfrak{S}_n$).

Example 3.3 ([M, §I.2 and §I.3]) Let $\{x_i\}$ be the set of indeterminates.

$$H(t) := \prod_{i=1}^{n} \frac{1}{1 - x_i t}$$

$$\Rightarrow \begin{cases} h_r & : \text{complete symmetric function of degree } r \\ e_r & : \text{elementary symmetric function of degree } r \\ p_r & = \sum_{i=1}^{n} x_i^r \end{cases}$$

$$\begin{aligned} s_\lambda &= \frac{a_{\lambda+\delta}}{a_\delta} \quad : \text{Schur function} \\ &= \frac{\sum_{w \in \mathfrak{S}_n} \varepsilon(w) w(x^{\lambda+\delta})}{\sum_{w \in \mathfrak{S}_n} \varepsilon(w) w(x^{\delta})} \ (\delta = (n-1, n-2, \cdots, 0)) \end{aligned}$$

Example 3.4 ([M, §I.2 Ex.3 and §I.3 Ex.1]) Let q be an indeterminates.

$$H(t) := \prod_{i=0}^{n-1} \frac{1}{1 - q^i t}$$

$$\Rightarrow h_r = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}, \quad e_r = q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix}, \quad p_r = \frac{1 - q^{nr}}{1 - q^r}$$

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{\prod_{j=1}^{r} (1 - q^{n-j+1})}{\prod_{i=1}^{r} (1 - q^i)} \quad : q\text{-binomial coefficient}$$

$$s_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}} \begin{cases} c(x) & : \text{content} \\ h(x) & : \text{hook length} \qquad \text{(See Example 2.4)} \\ n(\lambda) & = \sum_{i \geq 1} (i-1)\lambda_i \end{cases}$$

Example 3.5 ([M, §I.2 Ex.4 and §I.3 Ex.2]) Let $n \to \infty$ in Example 3.4.

$$H(t) := \prod_{i \geq 0} \frac{1}{1 - q^i t}$$

$$\Rightarrow h_r = \prod_{i=1}^{r} \frac{1}{1 - q^i}, \quad e_r = q^{r(r-1)/2} h_r, \quad p_r = \frac{1}{1 - q^r}$$

$$s_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1}{1 - q^{h(x)}}$$

Example 3.6 ([M, §I.2 Ex.5 and §I.3 Ex.3] and [Andrews, Chap.II])
Let a and b be indeterminates.

$$H(t) := \prod_{i\geq 0} \frac{1-bq^i t}{1-aq^i t}$$

$$\Rightarrow h_r = \prod_{i=1}^{r} \frac{a-bq^{i-1}}{1-q^i}, \quad e_r = \prod_{i=1}^{r} \frac{aq^{i-1}-b}{1-q^i}, \quad p_r = \frac{a^r-b^r}{1-q^r}$$

$$s_\lambda = q^{n(\lambda)} \prod_{x\in\lambda} \frac{a-bq^{c(x)}}{1-q^{h(x)}}$$

Proof for the last statement on s_λ:
Substitute $t \to t/a$ and we have $s_\lambda/a^{|\lambda|}$ instead of s_λ. So we can assume $a=1$. Next, note that the both hand sides of the equation to be proved are polynomials in b. If $b=q^n$ $(n=1,2,\cdots)$, it reduces to Example 3.5, hence the equation is valid for infinitely many value of b. We are done.

In the above Example 3.6, substitute $a=1$ and $b=-y$.
For $\sigma=(1,2,\cdots,r)\in\mathfrak{S}_r$: cyclic permutation, we have

$$\det(1-q\sigma|\mathbb{C}^r) = 1-q^r.$$

$$\Longrightarrow \quad p_r = \frac{1-(-y)^r}{1-q^r} = \frac{\det(1+y\sigma|\mathbb{C}^r)}{\det(1-q\sigma|\mathbb{C}^r)}$$

$\Longrightarrow$ For $w\in\mathfrak{S}_n$ with cyclic type μ;

$$p_\mu = \frac{\det(1+yw|\mathbb{C}^n)}{\det(1-qw|\mathbb{C}^n)} \; (=\tilde{\tau}(q,y)(w))$$

$$\begin{aligned}
s_\lambda &= q^{n(\lambda)} \prod_{x\in\lambda} \frac{1+yq^{c(x)}}{1-q^{h(x)}} \text{ (by Example 3.6)} \\
&= \sum_\mu \frac{1}{z_\mu}\chi^\lambda(\mu)p_\mu \text{ (by Theorem 3.2)} \\
&= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \chi^\lambda(w)p_{\mu(w)} \; (\mu(w) : \text{cyclic type of } w) \\
&= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \chi^\lambda(w)\tilde{\tau}(q,y)(w) \text{ (by the above)} \\
&= \langle \chi, \tilde{\tau}(q,y)\rangle_{\mathfrak{S}_n} = \tilde{\tau}(\chi;q,y)
\end{aligned}$$

4 Comments and references

Mathiew :
$\mathcal{H} := \mathbb{C}^l \backslash \{z_i = z_j\}$
cohomology $H^i(\mathcal{H})$: $\dim H^i(\mathcal{H}) = \#\{w \in \mathfrak{S}_l |\ l'(w) = i\}$
Is there any relation?
(Cf. Brieskorn [Brieskorn] and Orlik [Orlik].)

Schiffmann :
Is there a formula for $\kappa(\sigma)$?

$$\tau_1(\sigma; q, y) = \prod_{i=1}^{\kappa(\sigma)} (1 + q^{c_i} y)$$

partial answer:

$$\begin{cases} \text{type } A_{l-1} : & \kappa(\sigma) = l \\ \text{type } E_l \ \& \ \#(\text{family}) = 3 : & \kappa(\sigma) = l - 2 \\ \text{etc.} \end{cases}$$

Tanisaki :
How about for Hecke algebra $H_q(W)$?

$$L'(\sigma; t) := \sum_{w \in W} \sigma(T_w) t^{l'(w)}$$

Oshima :
$l'(w)$ can be expressed as a linear sum of characters. Is there a closed formula indicating this fact?
partial answer:

$$c(\chi; t) := \sum_{w \in W} \chi(w) t^{l'(w)} \Rightarrow \left.\frac{d}{dt} c(\chi; t)\right|_{t=1} = \#W \cdot \langle \chi, l' \rangle_W$$

Since $l'(w) = \sum_\chi \langle \chi, l' \rangle_W \chi(w)$, we have

$$l'(w) = \frac{1}{\#W} \sum_\chi \left(\left.\frac{d}{dt} c(\chi; t)\right|_{t=1} \right) \chi(w) = \left.\frac{d}{dt} \left(\frac{1}{\#W} \sum_\chi c(\chi; t) \chi(w) \right)\right|_{t=1}.$$

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