Hecke module structure on the orbits in double flag varieties

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REPRESENTATION THEORY XVII
02 OCT 2022 – 09 OCT 2022

Abstract

Let G be a connected reductive algebraic group and K its symmetric subgroup. We consider a double flag variety of finite type $\mathfrak{X} = K/B_K \times G/P$, where B_K is a Borel subgroup of K, and P a parabolic subgroup of G. The orbit space $\mathbb{C}\mathfrak{X}/K$ enjoys a natural Hecke module structure for the Hecke algebra $\mathscr{H} = \mathscr{H}(K, B_K)$ of K. However, it is difficult to find out its explicit Hecke module structure. In this talk, we consider the double flag variety of type AIII, i.e., when $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q \;\; (n = p + q)$, and give an explicit action of $\mathscr{H}$ on $\mathbb{C}\mathfrak{X}/K$ in combinatorial way using graphs.
The talk is based on the on-going joint work with Lucas Fresse in Université de Lorraine, IECL (France): `arXiv:2206.10476`.

Introduction

We focus on a double flag variety $\mathfrak{X} = K/B_K \times G/P$ of finite type:

- Geometry of orbits (dimensions, closure relations, conormal variety etc.)
- Combinatorics based on the graphs, Young tableaux, and RSK corresp
- Representations of Hecke algebras, Weyl groups, & Springer-Steinberg theory

Complete results on the double flag variety of type AIII, or

$$\mathfrak{X} = \left(\mathscr{F}\ell(\mathbb{C}^p) \times \mathscr{F}\ell(\mathbb{C}^q)\right) \times \mathsf{Gr}_r(\mathbb{C}^{p+q}) \quad \curvearrowleft K = \mathrm{GL}_p \times \mathrm{GL}_q$$

based on the (on-going) joint w with Lucas Fresse (IECL, Univ. Lorraine, France)

- Lucas Fresse and Kyo Nishiyama, *Action of Hecke algebra on the double flag variety of type AIII*, `arXiv:2206.10476 [math.RT]`.
- ——, *A Generalization of Steinberg Theory and an Exotic Moment Map*, International Mathematics Research Notices (2020), rnaa080.
- ——, *Orbit embedding for double flag varieties and Steinberg map*, Contemp. Math. 768 (2021), 21–42.
- ——, *On generalized Steinberg theory for type AIII*, 2021, `arXiv:2103.08460`.

Double flag varieties and K-orbits

Consider:

- G : connected reductive algebraic group
- K : symmetric subgroup
- $P \subset G, \quad Q \subset K$: parabolic subgroups (psg)

The double flag variety $\mathfrak{X}$ is introduced in (N-Ochiai 2011 [8])

$$K \curvearrowright \mathfrak{X} = K/Q \times G/P$$

In general $\#\mathfrak{X}/K = \infty$, but
$\exists$ interesting cases where $\#\mathfrak{X}/K < \infty$, called finite type (We will assume this)

even $\exists$ classification (He-N-Ochiai-Oshima 2013 [5]) of $\mathfrak{X}$ of finite type
(when $P = B_G$ or $Q = B_K$, i.e., one of them is a Borel subgrp)

Note that $\mathfrak{X}/K \simeq Q \backslash G/P$ (preserving closure relations)

Double flag variety of type AIII I

Take $\mathbb{C}$ as a base field (for convenience)

In this talk, we will concentrate on the case of symmetric sp of type AIII:

- $G = \mathrm{GL}_n$: general linear group
- $K = \mathrm{GL}_p \times \mathrm{GL}_q$: block diagnal subgrp of G $\quad (p + q = n)$
- $P = P_{(r,n-r)}$: max psg in G (with 2 diag blocks of size r & $n - r$)
- $Q = B_K = B_p \times B_q$: Borel subgrp in K

So that

$$\mathfrak{X} = \mathrm{GL}_p/B_p \times \mathrm{GL}_q/B_q \times \mathrm{GL}_n/P_{(r,n-r)}$$
$$\simeq \big(\mathscr{F}\!\ell(V^+) \times \mathscr{F}\!\ell(V^-)\big) \times \mathrm{Gr}_r(V),$$
where

- $V = V^+ \oplus V^-$ $(V^+ = \mathbb{C}^p,\ V^- = \mathbb{C}^q)$ polar decomposition
- $\mathrm{Gr}_r(V)$: Grassmannian of r-dim subsp's of V
- $\mathscr{F}\!\ell(V^\pm)$: complete flag varieties

Double flag variety of type AIII II

Lemma

$$\#\mathfrak{X}/K < \infty, \quad \textit{i.e.,} \quad \mathfrak{X} \textit{ is of finite type}$$

Write $X = \mathrm{Gr}_r(V) = G/P_{(r,n-r)} \quad \rightsquigarrow \quad K \curvearrowright X$: spherical (i.e., $\#X/B_K < \infty$)

Lemma

$\mathfrak{X} = \big(\mathscr{F}\ell(V^+) \times \mathscr{F}\ell(V^-)\big) \times \mathrm{Gr}_r(V) \quad \& \quad X = \mathrm{Gr}_r(V)$

$$\begin{array}{ccc} \mathfrak{X}/K & \xrightarrow{\simeq} & X/B_K \\ \Cup & & \Cup \\ K \cdot (\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau]) & \longmapsto & B_K \cdot [\tau] \end{array}$$

where $[\tau] \in \mathrm{Gr}_r(V) \quad \& \quad \mathcal{F}_0^\pm$*: standard flags stabilized by* B_p *or* B_q

We will often identify $\quad \mathfrak{X}/K \simeq X/B_K$

Let us describe what is the representative $\{\tau\}$

Description of K orbits on $\mathfrak{X}$

Partial permutation: $\tau_1 \in \mathfrak{T}_{p,r} \subset \mathrm{M}_{p,r}$

with entries of 0 or 1, in which $\#1 \leqslant 1$ in $\forall$rows and columns

$$\mathfrak{T} = \mathfrak{T}_{(p,q),r} := \left\{ \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathfrak{T}_{p,r} \times \mathfrak{T}_{q,r} \;\middle|\; \operatorname{rank} \tau = r \right\} \subset \mathrm{M}_{p+q,r}$$

: pairs of partial permutations of full rank

$\mathfrak{T} \curvearrowleft S_r \quad \rightsquigarrow \overline{\mathfrak{T}} = \mathfrak{T}/S_r$: quotient by the symmetric group action

Denote $[\tau] := \operatorname{Im} \tau \in \mathrm{Gr}_r(V)$: r-dim subsp gented by column vectors of τ

Lemma

$\mathfrak{T} \ni \tau \mapsto [\tau] \in \mathrm{Gr}_r(V)$ *factors through to*

$$\overline{\mathfrak{T}} = \mathfrak{T}/S_r \xrightarrow{\simeq} X/B_K \simeq \mathfrak{X}/K$$

so that we get $\overline{\mathfrak{T}} \simeq \mathfrak{X}/K$

$\exists$ convenient presentation by graphs...

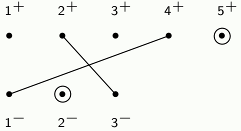
Graphs I

Represent $\tau \in \overline{\mathfrak{T}}$ by a graph $\Gamma(\tau)$:

- signed Vertices: positive vertices $\mathcal{V}_p^+ = \{1^+, \dots, p^+\}$ & negative vertices $\mathcal{V}_q^- = \{1^-, \dots, q^-\}$
- Edges between $i^+ \in \mathcal{V}_p^+$ and $j^- \in \mathcal{V}_q^-$ if τ contains two 1's at the positions i^+ & j^- in the same column
- Marking at the vertex i^+ or j^-, with only one 1 at i^+ or j^- in a column
- $\#(\text{Edges}) + \#(\text{Marked points}) = r$

Example ($(p, q) = (5, 3)$ and $r = 4$)

$$\tau = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc} \boldsymbol{e}_2^+ & \boldsymbol{e}_4^+ & \boldsymbol{e}_5^+ & 0 \\ \hline \boldsymbol{e}_3^- & \boldsymbol{e}_1^- & 0 & \boldsymbol{e}_2^- \end{array}\right) \rightsquigarrow \Gamma(\tau) =$$



Graphs II

Lemma

- $\mathcal{G}((p,q),r)$: *graphs with vertices* $\mathcal{V}_p^+ \cup \mathcal{V}_q^-$ *&*
  exactly r *edges & marked points*
- $\overline{\mathfrak{T}} = \mathfrak{T}/S_r$: *partial permutations*

$\rightsquigarrow$ *The graphs classify* K *orbits in* $\mathfrak{X}$

$$\begin{array}{ccccc} \mathfrak{X}/K \simeq X/B_K & \xleftarrow{\ \simeq\ } & \overline{\mathfrak{T}} & \xrightarrow{\ \simeq\ } & \mathcal{G}((p,q),r) \\ \Uplus & & \Uplus & & \Uplus \\ B_K \cdot [\tau] & \longleftarrow\!\dashv & \tau & \longmapsto & \Gamma(\tau) \end{array}$$

Orbital invariants: $a^{\pm}(\tau)$, $b(\tau)$, $c(\tau)$ & $R(\tau) = (r_{i,j}(\tau))$

For the graph $\Gamma(\tau)$ we define:

- degree of vertices: $\deg i^{\pm} := 0, 1, 2$ (NO [edges/marks], edges, marked)
- $a^{\pm}(\tau) := \#\{(i^{\pm}, j^{\pm}) \mid i < j \ \& \ \deg(i^{\pm}) < \deg(j^{\pm})\}$
- $b(\tau) := \#\{\text{edges}\}$ $c(\tau) := \#\{\text{crossings}\}$
- $r_{i,j}(\tau) := \#\text{Edges} + \#\text{Marks}$
  within vertices among $1^+ \leqslant k^+ \leqslant i^+$ & $1^- \leqslant \ell^- \leqslant j^-$

  $R(\tau) := (r_{i,j}(\tau))_{0 \leqslant i \leqslant p,\, 0 \leqslant j \leqslant q} \in \mathrm{M}_{p+1,q+1}$: the rank matrix

- decomposition $\mathcal{V}_p^+ = \{1, \ldots, p\} = I \sqcup L \sqcup L'$:

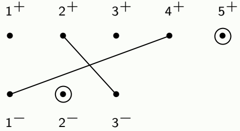
  $\{i \in \{1, \ldots, p\} \mid \ i^+$ is a vertex of degree 1 (resp. 2, 0)$\}$
- Similar decomp $\mathcal{V}_q^- = \{1, \ldots, q\} = J \sqcup M \sqcup M'$:

  $\{j \in \{1, \ldots, q\} \mid \ j^-$ is a vertex of degree 1 (resp. 2, 0)$\}$
- $\sigma : J \to I$: bijection defined by $\sigma(j) = i$ if (i^+, j^-) is an edge in $\Gamma(\tau)$.

Example

Let τ be as in (4):

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \left(\begin{array}{cccc} \boldsymbol{e}_2^+ & \boldsymbol{e}_4^+ & \boldsymbol{e}_5^+ & 0 \\ \hline \boldsymbol{e}_3^- & \boldsymbol{e}_1^- & 0 & \boldsymbol{e}_2^- \end{array} \right) \rightsquigarrow \Gamma(\tau) =$$



th

$$a^+(\tau) = 7, \quad a^-(\tau) = 1, \quad b(\tau) = 2, \quad c(\tau) = 1, \quad R(\tau) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$I = \{2,4\}, \quad L = \{5\}, \quad L' = \{1,3\},$$

$$J = \{1,3\}, \quad M = \{2\}, \quad M' = \varnothing, \quad \sigma = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in \mathrm{Bij}(J, I).$$

Dimensions and closure relations of Orbits

Recall the based point $(\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau])$ in $\mathfrak{X} = \big(\mathscr{F}\!\ell(V^+)\times\mathscr{F}\!\ell(V^-)\big) \times \mathrm{Gr}_r(V)$

Theorem

Denote a K orbit in $\mathfrak{X}$ by $\mathbb{O}_\tau := K \cdot (\mathcal{F}_0^+, \mathcal{F}_0^-, [\tau])$

1. $\dim \mathbb{O}_\tau = \dfrac{p(p-1)}{2} + \dfrac{q(q-1)}{2} + a^+(\tau) + a^-(\tau) + \dfrac{b(\tau)(b(\tau)+1)}{2} + c(\tau)$
2. $\mathbb{O}_\tau = \{(\mathcal{F}^+, \mathcal{F}^-, W) \mid \dim W \cap (\mathcal{F}_i^+ + \mathcal{F}_j^-) = r_{i,j}(\tau)$
   $\forall (i,j) \in \{0,\ldots,p\} \times \{0,\ldots,q\}\}$
3. $\overline{\mathbb{O}_\tau} \subset \overline{\mathbb{O}_{\tau'}} \iff r_{i,j}(\tau) \geqslant r_{i,j}(\tau') \ \ \forall (i,j) \in \{0,\ldots,p\} \times \{0,\ldots,q\}$

Can describe the cover relation of closure of orbits, which is not given today

See our recent preprint Fresse-N `arXiv:2103.08460` [3] for details.

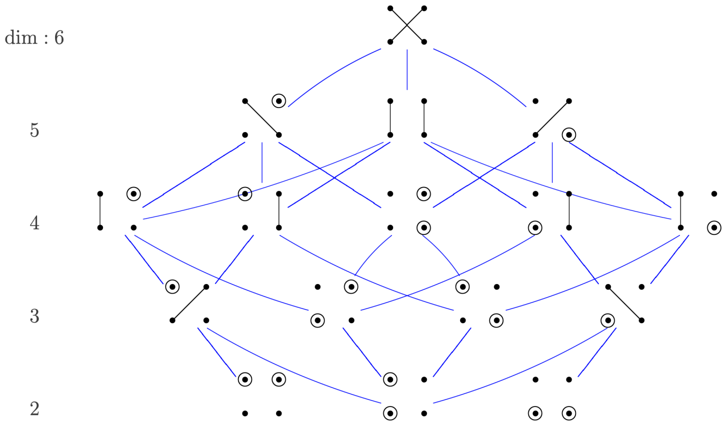
However, we notice

Corollary

$\mathbb{O}_{\tau'}$ *covers* $\mathbb{O}_\tau \implies \dim \mathbb{O}_{\tau'} = \dim \mathbb{O}_\tau + 1$

Example: Hasse Diagram

Figure: Closure relations of K orbits for $p = q = r = 2$



Hecke algebra action

$\exists$ standard way to define Hecke algebra action of $\mathscr{H} = \mathscr{H}(K, B_K)$ on the DFV $\mathfrak{X}$ by the convolution product (cf. Chriss-Ginzburg [1, § 2.7]):

$$\begin{array}{ccc} & K/B_K \times K/B_K \times G/P & \\ {}^{p_{12}}\swarrow & & \searrow^{p_{23}} \\ K/B_K \times K/B_K & & K/B_K \times G/P = \mathfrak{X} \end{array}$$

However, we prefer a simpler picture

$$\begin{array}{ccc} & K \times_{B_K} G/P & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ K/B_K & & G/P = X \end{array}$$

More generally, if X is a spherical K-variety, Hecke algebra/Weyl group actions are considered by Mars-Springer [7] and Knop [6].

Orbit multiplications

As in the above, $\tau \in \mathfrak{T}$ is identified with a graph with vertices $\mathcal{V}_p^+ \sqcup \mathcal{V}_q^-$ which are equipped with several markings and edges.

Lemma

A double coset $B_K s_i B_K$ generates at most two B_K orbits on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{F}^n)$. Namely we have

$$B_K s_i B_K \cdot \tau = \begin{cases} B_K s_i \tau = B_K \tau & \text{if } s_i \tau = \tau \qquad \text{case (I)} \\ B_K s_i \tau \cup B_K \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is in case (II)} \\ B_K s_i \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is in case (III)} \end{cases}$$

where

- case (I): $i, i+1$ *are both of degree 0 (isolated) or both of degree 2 (marked)*
- case (II): $\deg_\tau(i) < \deg_\tau(i+1)$ *or* $i, i+1$ *are end points of edges with crossing*
- case (III): $\deg_\tau(i) > \deg_\tau(i+1)$ *or* $i, i+1$ *are end points of edges without crossing*

Hecke algebra actions

Theorem

The action of the generators in $\mathscr{H} = \mathscr{H}(K, B_K)$ is given by ($\boldsymbol{q}$: indeterminate)

$$T_i * \xi_\tau = \begin{cases} \boldsymbol{q}\,\xi_\tau & (s_i\tau = \tau) & \text{Case (I)} \\ (\boldsymbol{q}-1)\,\xi_\tau + \boldsymbol{q}\,\xi_{s_i\tau} & (s_i\tau \neq \tau) & \text{Case (II)} \\ \xi_{s_i\tau} & (s_i\tau \neq \tau) & \text{Case (III)} \end{cases}$$

they satisfy the Hecke algebra relations:

$$(T_s + 1)(T_s - \boldsymbol{q}) = 0 \tag{3.1}$$

$$T_{ww'} = T_w T_{w'} \qquad \text{if } \ell(ww') = \ell(w) + \ell(w') \tag{3.2}$$

If we specialize $\boldsymbol{q} = 1$, then the Hecke alg rep goes down to that of the Weyl group $W_K = S_p \times S_q$. It's just a permutation on the vertices of the graphs. Thus we get

Weyl group representations

Corollary

The representation of $W_K = S_p \times S_q$ *on* $\mathbb{C}[\mathfrak{X}/K]$ *is isomorphic to*

$$\bigoplus_{(k,s,t)} \operatorname{Ind}_{H_{k,s,t}}^{S_p \times S_q} \mathbf{1}$$

where (k, s, t) *moves over*

$$p \geqslant k + s, \quad q \geqslant k + t, \quad r = k + s + t$$

and

$$H_{k,s,t} \simeq \Delta S_k \times S_s \times S_{s'} \times S_t \times S_{t'},$$

with $s' = p - (k + s)$ *&* $t' = q - (k + t)$.

We can describe the basis of the representation by pairs of special Young tableaux through generalized/exotic Steinberg map

(Fresse-N, 2020 IMRN and Contemp. Math. [2, 4]).

$\rightsquigarrow$ Springer type theorem associated to nilpotent orbits

Generalized/Exotic Steinberg theory I

Recall the double flag variety (in a broader context)

$$\mathfrak{X} = \mathscr{B}_K \times \mathscr{P}_G, \qquad \mathscr{B}_K = K/B_K,\ \mathscr{P}_G = G/P$$

Identify $\mathscr{P}_G = G/P$ with the set of parabolic subalgs $\mathfrak{p}_1 \overset{conj}{\sim} \mathfrak{p} = \mathrm{Lie}\,(P)$
Then the cotangent bundle over $\mathscr{P}_G$:

$$T^*\mathscr{P}_G = \{(\mathfrak{p}_1, x) \mid \mathfrak{p}_1 \in \mathscr{P}_G,\ x \in \mathfrak{u}_{\mathfrak{p}_1}\} \simeq G \times_P \mathfrak{u}_{\mathfrak{p}},$$

where $\mathfrak{u}_{\mathfrak{p}}$ = nilradical($\mathfrak{p}$)

The moment map:

$$\begin{array}{cccc} \mu_{\mathscr{P}_G} : & T^*\mathscr{P}_G & \longrightarrow & \mathcal{N}_{\mathfrak{g}} \\ & \Downarrow & & \Downarrow \\ & (\mathfrak{p}_1, x) & \longmapsto & x \end{array} \quad = (\text{nilpotent variety}) \quad (\text{2nd proj})$$

wrt a standard symplectic structure on $T^*\mathscr{P}_G$.
Similarly, we have the moment map

$$\mu_{\mathscr{B}_K} : T^*\mathscr{B}_K = \{(\mathfrak{q}_1, y) \mid \mathfrak{q}_1 \in \mathscr{B}_K,\ y \in \mathfrak{u}_{\mathfrak{q}_1}\} \to \mathcal{N}_{\mathfrak{k}} = (\text{nilpotent var for symm sp}),$$

$$\mu_{\mathscr{B}_K}(\mathfrak{q}_1, y) = y$$

Generalized/Exotic Steinberg theory II

Definition

$\mathcal{Y} := T^*\mathscr{B}_K \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathscr{P}_G$: fiber product over nilpotent var $\mathcal{N}_{\mathfrak{k}}$:

$$
\begin{array}{ccccc}
\mathcal{Y} = T^*\mathscr{B}_K \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathscr{P}_G & & \xrightarrow{\quad p_1 \quad} & & T^*\mathscr{P}_G \ni (\mathfrak{p}_1, x) \\
 & & & & \downarrow{\scriptstyle \mu_{\mathscr{P}_G}} \\
\downarrow{\scriptstyle p_2} & & \searrow{\scriptstyle \varphi^\theta} & & \mathcal{N}_{\mathfrak{g}} \ni x \\
 & & & & \downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{k}}} \\
(\mathfrak{q}_1, y) \in T^*\mathscr{B}_K & & \xrightarrow{\quad -\mu_{\mathscr{B}_K} \quad} & & \mathcal{N}_{\mathfrak{k}} \ni -y = x^\theta
\end{array}
$$

We call $\mathcal{Y} = \mathcal{Y}_{\mathfrak{X}}$ the conormal variety for the double flag variety $\mathfrak{X}$.

Generalized/Exotic Steinberg theory III

Fact

1. *Let* $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \to \mathfrak{k}$ *be the moment map on* $T^*\mathfrak{X}$.
   Then $\mathcal{Y} \simeq \mu_{\mathfrak{X}}^{-1}(0)$ *is the null fiber.*
2. $\mathcal{Y}$ *is a disjoint union of the conormal bundles:* $\mathcal{Y} = \coprod_{\mathbb{O} \in \mathfrak{X}/K} T^*_{\mathbb{O}}\mathfrak{X}$
3. $\mathcal{Y} = \bigcup_{\mathbb{O} \in \mathfrak{X}/K} \overline{T^*_{\mathbb{O}}\mathfrak{X}}$ *gives the irred decomp*

the diagonal map in the fiber product: $\varphi^\theta : \mathcal{Y} \to \mathcal{N}_{\mathfrak{k}}$:

$$\varphi^\theta((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^\theta = -y \quad \text{for} \quad ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

we need another map

$$\varphi^{-\theta}((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^{-\theta} = x + y \quad \text{for} \quad ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

- φ^θ the generalized Steinberg map and $\operatorname{Im} \varphi^\theta \subset \mathcal{N}_{\mathfrak{k}}$
- $\varphi^{-\theta}$ the exotic Steinberg map and $\operatorname{Im} \varphi^{-\theta} \subset \mathcal{N}_{\mathfrak{s}}$

Generalized/Exotic Steinberg theory IV

$\pi : T^*\mathfrak{X} \to \mathfrak{X}$: bundle map $\rightsquigarrow$ K-equiv double fibration

$$\begin{array}{ccc} & \mathcal{Y} = T^*\mathscr{B}_K \times_{\mathcal{N}^\theta} T^*\mathscr{P}_G & \\ \swarrow^{\pi} & & \searrow^{\varphi^{\pm\theta}} \\ \mathfrak{X} = \mathscr{B}_K \times \mathscr{P}_G & & \mathcal{N}^{\pm\theta} \end{array}$$

Here $\mathcal{N}^\theta = \mathcal{N}_\mathfrak{k}$, $\mathcal{N}^{-\theta} = \mathcal{N}_\mathfrak{s}$. Using this diagram, we define orbit maps

$$\begin{array}{ccc} \Phi^{\pm\theta} : \mathfrak{X}/K & \longrightarrow & \mathcal{N}^{\pm\theta}/K \\ \Cup & & \Cup \\ \mathbb{O} & \longmapsto & \mathcal{O} \end{array} \qquad \text{by} \qquad \overline{\varphi^{\pm\theta}(\pi^{-1}(\mathbb{O}))} = \overline{\mathcal{O}},$$

where $\mathbb{O}$ is a K-orbit in $\mathfrak{X}$ and $\mathcal{O}$ is a nilpotent K-orbit in $\mathcal{N}^{\pm\theta}$.

Since $\pi^{-1}(\mathbb{O}) = T^*_{\mathbb{O}}\mathfrak{X}$, $\qquad \Phi^{\pm\theta}(\mathbb{O}) = \mathcal{O} \iff \overline{\varphi^{\pm\theta}(T^*_{\mathbb{O}}\mathfrak{X})} = \overline{\mathcal{O}}$.

Thus $\Phi^{\pm\theta}$ is a map from irred compnts of $\mathcal{Y}$ to nilpotent K orbits (in $\mathfrak{k}$ or $\mathfrak{s}$).

Generalized/Exotic Steinberg theory V

$\pi : T^*\mathfrak{X} \to \mathfrak{X}$: bundle map $\rightsquigarrow$ K-equiv double fibration

$$\begin{array}{ccc} & \mathcal{Y} = T^*\mathscr{B}_K \times_{\mathcal{N}^\theta} T^*\mathscr{P}_G & \\ \swarrow^{\pi} & & \searrow^{\varphi^{\pm\theta}} \\ \mathfrak{X} = \mathscr{B}_K \times \mathscr{P}_G & & \mathcal{N}^{\pm\theta} \end{array}$$

Here $\mathcal{N}^\theta = \mathcal{N}_{\mathfrak{k}}$, $\mathcal{N}^{-\theta} = \mathcal{N}_{\mathfrak{s}}$. Using this diagram, we define orbit maps

$$\begin{array}{ccc} \Phi^{\pm\theta} : \mathfrak{X}/K & \longrightarrow & \mathcal{N}^{\pm\theta}/K \\ \Uplus & & \Uplus \\ \mathbb{O} & \longmapsto & \mathcal{O} \end{array} \qquad \text{by} \qquad \overline{\varphi^{\pm\theta}(\pi^{-1}(\mathbb{O}))} = \overline{\mathcal{O}},$$

where $\mathbb{O}$ is a K-orbit in $\mathfrak{X}$ and $\mathcal{O}$ is a nilpotent K-orbit in $\mathcal{N}^{\pm\theta}$.

- Φ^θ called the generalized Steinberg map and
- $\Phi^{-\theta}$ called the exotic Steinberg map.

Generalized RS correspondence for type AIII

Recall $\overline{\mathfrak{T}} = \mathfrak{T}/S_r \simeq \mathfrak{X}/K$: pairs of partial permutations classifying K orbits on $\mathfrak{X}$.

$\exists$ generalized RS correspondence to pairs of standard tableaux with decorations.

Notation

Define $\lambda' \Subset \lambda \iff \lambda' \subset \lambda$ & the skew tableau λ/λ' is *column strip*

$\mathscr{P}(n) := \{\lambda \vdash n\}$: the set of partitions of n

Theorem (gen RS correspondence)

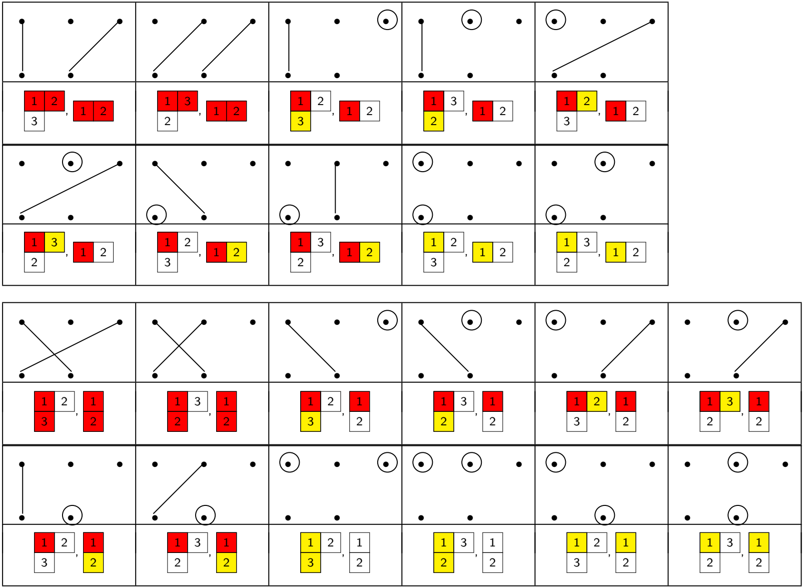
$\exists$ combinatorial bijection between the pairs of partial permutations and pairs of decorated standard tableaux: $\overline{\mathfrak{T}} \xrightarrow{\simeq} \bigsqcup_{(\lambda,\mu)\in\mathscr{P}(p)\times\mathscr{P}(q)} \mathcal{T}_{\lambda,\mu}$

where $\mathcal{T}_{\lambda,\mu} = \{(T_1, T_2; \lambda', \mu'; \nu)$ satisfying $(*)$ & $(**)$ below$\}$

$(*)$ $(T_1, T_2) \in \mathrm{STab}(\lambda) \times \mathrm{STab}(\mu)$

$(**)$ $\nu \Subset \lambda' \Subset \lambda$, $\nu \Subset \mu' \Subset \mu$ & $|\lambda'| + |\mu'| = |\nu| + r$.

Examples of gen RSK corr: $(p, q, r) = (3, 2, 2)$ ([3])



Get a commutative diagram

$$\begin{array}{ccccc} \mathfrak{X}/K \simeq \mathfrak{T}_{(p,q;r)}/S_r & \xrightarrow[\text{genRS}]{\sim} & \displaystyle\coprod_{(\lambda,\mu)\in\mathscr{P}(p)\times\mathscr{P}(q)} \mathcal{T}_{\lambda,\mu} & \ni & (T_1, T_2; \lambda', \mu'; \nu) \\ & \searrow^{\Phi^\theta} & \downarrow & & \downarrow \\ & & \mathscr{P}(p)\times\mathscr{P}(q) & \ni & (\lambda, \mu) \end{array}$$

This diagram actually commutes with the Weyl group representations i.e.,

> the fiber of a nilpotent orbit $\mathcal{O}_{(\lambda,\mu)} \subset \mathcal{N}_{\mathfrak{k}}$ inherits a structure of Weyl group representations (Springer correspondence)

However, we don't know a rigorous geometric reason of this phenomenon

- Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR MR1433132 (98i:22021)

- Lucas Fresse and Kyo Nishiyama, *A Generalization of Steinberg Theory and an Exotic Moment Map*, International Mathematics Research Notices (2020), rnaa080.

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