Functional equation of an enhanced zeta distribution — the case of positive symmetric cone

joint work in progress with Bent Ørsted & Akihito Wachi

Kyo Nishiyama　西山 享

Aoyama Gakuin University　青山学院大学

WORKSHOP "Dirac operators and representation theory"
Zagreb University, June 18-22, 2018

Plan of talk

1. Zeta distributions/integrals
   - Classical examples of zeta integrals
   - Prehomogeneous vector spaces
   - Fundamental Theorem

Plan of talk

1. Zeta distributions/integrals
    - Classical examples of zeta integrals
    - Prehomogeneous vector spaces
    - Fundamental Theorem
2. Enhanced zeta integral and its meromorphic continuation
    - b-functions
    - gamma factors and meromorphic continuation

Plan of talk

1. Zeta distributions/integrals
   - Classical examples of zeta integrals
   - Prehomogeneous vector spaces
   - Fundamental Theorem
2. Enhanced zeta integral and its meromorphic continuation
   - b-functions
   - gamma factors and meromorphic continuation
3. Fourier transform and a functional equation
   - Fourier transform of kernel function
   - Functional equation

Plan of talk

1. Zeta distributions/integrals
   - Classical examples of zeta integrals
   - Prehomogeneous vector spaces
   - Fundamental Theorem
2. Enhanced zeta integral and its meromorphic continuation
   - b-functions
   - gamma factors and meromorphic continuation
3. Fourier transform and a functional equation
   - Fourier transform of kernel function
   - Functional equation
4. Idea of Proof, Further problems
   - Idea of Proof
   - Zeta integrals associated with orbits

Plan of talk

1. Zeta distributions/integrals
    - Classical examples of zeta integrals
    - Prehomogeneous vector spaces
    - Fundamental Theorem
2. Enhanced zeta integral and its meromorphic continuation
    - *b*-functions
    - gamma factors and meromorphic continuation
3. Fourier transform and a functional equation
    - Fourier transform of kernel function
    - Functional equation
4. Idea of Proof, Further problems
    - Idea of Proof
    - Zeta integrals associated with orbits
5. Motivations

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

Tate's zeta integral

$$Z^{\mathrm{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$
- Meromorphic continuation to $\mathbb{C}$

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$
- Meromorphic continuation to $\mathbb{C}$
- Functional equation:

$$Z^{\mathsf{T}}(\varphi, s) = \frac{\pi^{-s-\frac{1}{2}}\Gamma((s+1)/2)}{\Gamma(-s/2)} Z^{\mathsf{T}}(\widehat{\varphi}, -s-1)$$

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$
- Meromorphic continuation to $\mathbb{C}$
- Functional equation:

$$Z^{\mathsf{T}}(\varphi, s) = \frac{\pi^{-s-\frac{1}{2}}\Gamma((s+1)/2)}{\Gamma(-s/2)} Z^{\mathsf{T}}(\widehat{\varphi}, -s-1)$$

$\cdots$

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$
- Meromorphic continuation to $\mathbb{C}$
- Functional equation:

$$Z^{\mathsf{T}}(\varphi, s) = \frac{\pi^{-s-\frac{1}{2}}\Gamma((s+1)/2)}{\Gamma(-s/2)} Z^{\mathsf{T}}(\widehat{\varphi}, -s-1)$$

$\cdots$ just copied it from Leticia's paper (JFA 2004) [Bar04]

Tate's zeta integral

$$Z^{\mathsf{T}}(\varphi, s) = \int_{\mathbb{R}} \varphi(z)|z|^s dz \qquad (s \in \mathbb{C},\ \varphi \in \mathscr{S}(\mathbb{R}))$$

- Convergence: $\operatorname{Re} s > -1$
- Meromorphic continuation to $\mathbb{C}$
- Functional equation:

$$Z^{\mathsf{T}}(\varphi, s) = \frac{\pi^{-s-\frac{1}{2}}\Gamma((s+1)/2)}{\Gamma(-s/2)} Z^{\mathsf{T}}(\widehat{\varphi}, -s-1)$$

$\cdots$ just copied it from Leticia's paper (JFA 2004) [Bar04]

- Related to Riemann zeta $\zeta(s)$

Godement-Jacquet zeta integral

A generalization to $(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}), \mathrm{M}_n(\mathbb{R}))$:

$$Z^{\mathrm{GJ}}(\varphi, s) = \int_{\mathrm{M}_n(\mathbb{R})} \varphi(z) |\det z|^s dz$$

Again we have

Godement-Jacquet zeta integral

A generalization to $(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}), \mathrm{M}_n(\mathbb{R}))$:

$$Z^{\mathrm{GJ}}(\varphi, s) = \int_{\mathrm{M}_n(\mathbb{R})} \varphi(z) |\det z|^s dz$$

Again we have $\cdots$

Godement-Jacquet zeta integral

A generalization to $(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}), \mathrm{M}_n(\mathbb{R}))$:

$$Z^{\mathrm{GJ}}(\varphi, s) = \int_{\mathrm{M}_n(\mathbb{R})} \varphi(z) |\det z|^s dz$$

Again we have $\cdots$ copied it from Leticia's paper (JFA 2004) [Bar04]

- Convergence: $\mathrm{Re}\, s > -1$ with meromorphic continuation to $\mathbb{C}$
- Functional equation:
$$\frac{\pi^{\frac{1}{2}n(s+n)}}{\Gamma_n(\frac{s+n}{2})} Z^{\mathrm{GJ}}(\varphi, s) = \frac{\pi^{-\frac{n}{2}s}}{\Gamma_n(-s/2)} Z^{\mathrm{GJ}}(\widehat{\varphi}, -s-n)$$
- Related to zeta function $\zeta(L, s)$ for a lattice L $\cdots$ important in NT

Godement-Jacquet zeta integral

A generalization to $(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}), \mathrm{M}_n(\mathbb{R}))$:

$$Z^{\mathrm{GJ}}(\varphi, s) = \int_{\mathrm{M}_n(\mathbb{R})} \varphi(z) |\det z|^s dz$$

Again we have $\cdots$ copied it from Leticia's paper (JFA 2004) [Bar04]

- Convergence: $\operatorname{Re} s > -1$ with meromorphic continuation to $\mathbb{C}$
- Functional equation:
$$\frac{\pi^{\frac{1}{2}n(s+n)}}{\Gamma_n(\frac{s+n}{2})} Z^{\mathrm{GJ}}(\varphi, s) = \frac{\pi^{-\frac{n}{2}s}}{\Gamma_n(-s/2)} Z^{\mathrm{GJ}}(\widehat{\varphi}, -s-n)$$
- Related to zeta function $\zeta(L, s)$ for a lattice L $\cdots$ important in NT

$\exists$ further generalizat'n to $(\mathrm{GL}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}))$ (Shintani, Satake-Faraut, ...)

Godement-Jacquet zeta integral

A generalization to $(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}), \mathrm{M}_n(\mathbb{R}))$:

$$Z^{\mathrm{GJ}}(\varphi, s) = \int_{\mathrm{M}_n(\mathbb{R})} \varphi(z) |\det z|^s dz$$

Again we have $\cdots$ copied it from Leticia's paper (JFA 2004) [Bar04]

- Convergence: $\operatorname{Re} s > -1$ with meromorphic continuation to $\mathbb{C}$
- Functional equation:
$$\frac{\pi^{\frac{1}{2}n(s+n)}}{\Gamma_n(\frac{s+n}{2})} Z^{\mathrm{GJ}}(\varphi, s) = \frac{\pi^{-\frac{n}{2}s}}{\Gamma_n(-s/2)} Z^{\mathrm{GJ}}(\widehat{\varphi}, -s-n)$$
- Related to zeta function $\zeta(L, s)$ for a lattice L $\cdots$ important in NT

$\exists$ further generalizat'n to $(\mathrm{GL}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}))$ (Shintani, Satake-Faraut, ...)

Question : What is a right frame work?

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Setting (roughly):

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Setting (roughly):

1. (G, V) : PV $/\mathbb{C}$, i.e., $\exists$ an open orbit

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Setting (roughly):

1. (G, V) : PV $/\mathbb{C}$, i.e., $\exists$ an open orbit
2. $P(z) \in \mathbb{C}[V]$: fundamental relative invariant with character χ_P

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Setting (roughly):

1. (G, V) : PV $/\mathbb{C}$, i.e., $\exists$ an open orbit
2. $P(z) \in \mathbb{C}[V]$: fundamental relative invariant with character χ_P
3. $V_{\mathbb{R}} \backslash \{P = 0\} = \bigcup_{i=1}^{\ell} \mathcal{O}_i$: decomposition to open orbits

Prehomogeneous Vector Space

Frame work brought by Shintani and Mikio Sato ($\sim$ 70's, [SS74]):

Setting (roughly):

1. (G, V) : PV $/\mathbb{C}$, i.e., $\exists$ an open orbit
2. $P(z) \in \mathbb{C}[V]$: fundamental relative invariant with character χ_P
3. $V_{\mathbb{R}} \setminus \{P = 0\} = \bigcup_{i=1}^{\ell} \mathcal{O}_i$: decomposition to open orbits

Definition 2.1 (local zeta integral)

$$Z_i^{(G,V)}(\varphi, s) = \int_{\mathcal{O}_i} \varphi(z) |P(z)|^s dz$$

Fundamental Theorem of PV

Theorem 2.2 (Sato-Shintani [SS74])

Assume G is reductive and $\exists$ $P(z)$ only one fundamental rel inv

Fundamental Theorem of PV

Theorem 2.2 (Sato-Shintani [SS74])

Assume G is reductive and $\exists\ P(z)$ only one fundamental rel inv

1. $Z_i(\varphi, s)$ *converges in* $\operatorname{Re} s > 0$ *and continued meromorphically to* $\mathbb{C}$
   $\leftsquigarrow$ *b-function and Bernstein-Sato identity*

Fundamental Theorem of PV

Theorem 2.2 (Sato-Shintani [SS74])

Assume G is reductive and $\exists\ P(z)$ only one fundamental rel inv

1. $Z_i(\varphi, s)$ *converges in* $\operatorname{Re} s > 0$ *and continued meromorphically to* $\mathbb{C}$
   $\leftsquigarrow$ *b-function and Bernstein-Sato identity*
2. *local functional equation:* $n = \dim V, d = \deg P$

$$Z_i(\widehat{\varphi}, s - \tfrac{n}{d}) = \gamma(s - \tfrac{n}{d}) \sum_{i=1}^{\ell} u_{ij}(s) Z_j(\varphi, -s)$$

Fundamental Theorem of PV

Theorem 2.2 (Sato-Shintani [SS74])

Assume G is reductive and $\exists\ P(z)$ only one fundamental rel inv

1. $Z_i(\varphi, s)$ *converges in* $\operatorname{Re} s > 0$ *and continued meromorphically to* $\mathbb{C}$
   $\leftsquigarrow$ *b-function and Bernstein-Sato identity*
2. *local functional equation:* $n = \dim V, d = \deg P$

$$Z_i(\widehat{\varphi}, s - \tfrac{n}{d}) = \gamma(s - \tfrac{n}{d}) \sum_{i=1}^{\ell} u_{ij}(s) Z_j(\varphi, -s)$$

Remark 2.3

Gamma factor $\gamma(s)$ can be described explicitly in terms of b-function
$u_{ij}(s)$ is a product of exponential function and a polynomial in $e^{\pm\pi i s}$

Fundamental Theorem of PV

Theorem 2.2 (Sato-Shintani [SS74])

Assume G is reductive and $\exists\ P(z)$ only one fundamental rel inv

1. *$Z_i(\varphi, s)$ converges in $\operatorname{Re} s > 0$ and continued meromorphically to $\mathbb{C}$*
   $\leftsquigarrow$ *b-function and Bernstein-Sato identity*
2. *local functional equation: $n = \dim V$, $d = \deg P$*

$$Z_i(\widehat{\varphi}, s - \tfrac{n}{d}) = \gamma(s - \tfrac{n}{d}) \sum_{i=1}^{\ell} u_{ij}(s) Z_j(\varphi, -s)$$

Remark 2.3

Gamma factor $\gamma(s)$ can be described explicitly in terms of b-function
$u_{ij}(s)$ is a product of exponential function and a polynomial in $e^{\pm \pi i s}$

The Fundamental Theorem is generalized to the case of several complex variables by Fumihiro Sato [Sat82a] [Sat83] [Sat82b]

Enhanced space

Aim

- To investigate the zeta integral for PV
  with two fundamental relative invariants
- Generalization of existing results on $\mathrm{M}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}), \ldots$ etc

Enhanced space

Aim

- To investigate the zeta integral for PV
  with two fundamental relative invariants
- Generalization of existing results on $\mathrm{M}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}), \ldots$ etc

Setting:

- $V = \mathrm{Sym}_n(\mathbb{R})$: Euclidean Jordan algebra (cf. Faraut-Korányi [FK94])
  action by $L = \mathrm{Str}\,(V) = \mathrm{GL}_n(\mathbb{R})$

Enhanced space

Aim

- To investigate the zeta integral for PV
  with two fundamental relative invariants
- Generalization of existing results on $\mathsf{M}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}), \ldots$ etc

Setting:

- $V = \mathrm{Sym}_n(\mathbb{R})$: Euclidean Jordan algebra (cf. Faraut-Korányi [FK94])
  action by $L = \mathrm{Str}\,(V) = \mathsf{GL}_n(\mathbb{R})$
- $E = \mathsf{M}_{n,d}(\mathbb{R}) = \oplus^d \mathbb{R}^n$: representation of V
  d-copies of natural rep, action by $\mathrm{Str}\,(V) = \mathsf{GL}_n(\mathbb{R})$ & $H = \mathsf{GL}_d(\mathbb{R})$

Enhanced space

Aim

- To investigate the zeta integral for PV
  with two fundamental relative invariants
- Generalization of existing results on $\mathrm{M}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R}), \ldots$ etc

Setting:

- $V = \mathrm{Sym}_n(\mathbb{R})$: Euclidean Jordan algebra (cf. Faraut-Korányi [FK94])
  action by $L = \mathrm{Str}\,(V) = \mathrm{GL}_n(\mathbb{R})$
- $E = \mathrm{M}_{n,d}(\mathbb{R}) = \oplus^d \mathbb{R}^n$: representation of V
  d-copies of natural rep, action by $\mathrm{Str}\,(V) = \mathrm{GL}_n(\mathbb{R})$ & $H = \mathrm{GL}_d(\mathbb{R})$
- $W = V \oplus E = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{M}_{n,d}(\mathbb{R})$: enhanced space
  action of $G = L \times H = \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$ via

$$(g,h)\cdot(z,y) = (gz\,{}^t g, gy\,{}^t h) \quad \text{where} \quad \begin{array}{l}(g,h) \in L \times H = G \\ (z,y) \in V \oplus E = W\end{array}$$

Two relative invariants P_1, P_2

Extend the base field $\mathbb{R}$ to $\mathbb{C}$

$$G_{\mathbb{C}} \curvearrowright W_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{M}_{n,d}(\mathbb{C}) \text{: PV}$$

Two relative invariants P_1, P_2

Extend the base field $\mathbb{R}$ to $\mathbb{C}$

$$G_{\mathbb{C}} \curvearrowright W_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{M}_{n,d}(\mathbb{C}) \text{: PV}$$

Lemma 3.1

Assume $d \leqslant n$. Then there are two fundamental relative invts of $(G_{\mathbb{C}}, W_{\mathbb{C}})$: for $(z, y) \in W_{\mathbb{C}} = \mathrm{Sym}_n \times \mathrm{M}_{n,d}$, $(g, h) \in G_{\mathbb{C}}$

$$P_1(z, y) = \det z \qquad \text{with char } \chi_{P_1}(g, h) = (\det g)^2$$

$$P_2(z, y) = (-1)^d \det \begin{pmatrix} z & y \\ {}^t y & 0 \end{pmatrix} \qquad \text{with char } \chi_{P_2}(g, h) = (\det g)^2 (\det h)^2$$

$\forall$ relative invariants are of the form $P_1^{m_1} P_2^{m_2} (m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$

Two relative invariants P_1, P_2

Extend the base field $\mathbb{R}$ to $\mathbb{C}$

$$G_{\mathbb{C}} \curvearrowright W_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{M}_{n,d}(\mathbb{C}) \text{: PV}$$

Lemma 3.1

Assume $d \leqslant n$. Then there are two fundamental relative invts of $(G_{\mathbb{C}}, W_{\mathbb{C}})$: for $(z, y) \in W_{\mathbb{C}} = \mathrm{Sym}_n \times \mathrm{M}_{n,d}$, $(g, h) \in G_{\mathbb{C}}$

$$P_1(z, y) = \det z \qquad \textit{with char } \chi_{P_1}(g, h) = (\det g)^2$$

$$P_2(z, y) = (-1)^d \det \begin{pmatrix} z & y \\ {}^t y & 0 \end{pmatrix} \qquad \textit{with char } \chi_{P_2}(g, h) = (\det g)^2 (\det h)^2$$

$\forall$ *relative invariants are of the form* $P_1^{m_1} P_2^{m_2} (m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$

Note: $z \in \mathrm{Sym}_n(\mathbb{C})$ is regular $\implies P_2(z, y) = \det z \cdot \det\left({}^t y z^{-1} y \right)$

Two relative invariants P_1, P_2

Extend the base field $\mathbb{R}$ to $\mathbb{C}$

$$G_{\mathbb{C}} \curvearrowright W_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{M}_{n,d}(\mathbb{C}): \text{ PV}$$

Lemma 3.1

Assume $d \leqslant n$. Then there are two fundamental relative invts of $(G_{\mathbb{C}}, W_{\mathbb{C}})$: for $(z, y) \in W_{\mathbb{C}} = \mathrm{Sym}_n \times \mathrm{M}_{n,d}$, $(g, h) \in G_{\mathbb{C}}$

$$P_1(z, y) = \det z \quad \text{with char } \chi_{P_1}(g, h) = (\det g)^2$$

$$P_2(z, y) = (-1)^d \det \begin{pmatrix} z & y \\ {}^t y & 0 \end{pmatrix} \quad \text{with char } \chi_{P_2}(g, h) = (\det g)^2 (\det h)^2$$

$\forall$ *relative invariants are of the form* $P_1^{m_1} P_2^{m_2} (m_1, m_2 \in \mathbb{Z}_{\geqslant 0})$

Note: $z \in \mathrm{Sym}_n(\mathbb{C})$ is regular $\implies P_2(z, y) = \det z \cdot \det\left({}^t y z^{-1} y \right)$

If $\underline{d > n} \implies P_2 \equiv 0$ & only <u>one</u> rel inv P_1 survives

$\therefore$ We always assume $d \leqslant n$ below

b-functions

inner product in $W_{\mathbb{C}}$: $\langle (z, y), (w, x) \rangle = \operatorname{Tr} zw + \operatorname{Tr} {}^t y x$

b-functions

inner product in $W_{\mathbb{C}}$: $\langle (z, y), (w, x) \rangle = \mathrm{Tr}\, zw + \mathrm{Tr}\ {}^t yx$

$P_i^*(\partial_{z,y})$: const coeff diff op defined by

$$P_i^*(\partial_{z,y}) e^{\langle (z,y),(w,x) \rangle} = P_i(w, x) e^{\langle (z,y),(w,x) \rangle} \quad (z, w \in \mathrm{Sym}_n, y, x \in \mathrm{M}_{n,d})$$

b-functions

inner product in $W_{\mathbb{C}}$: $\langle (z,y),(w,x)\rangle = \operatorname{Tr} zw + \operatorname{Tr} {}^t yx$

$P_i^*(\partial_{z,y})$: const coeff diff op defined by

$$P_i^*(\partial_{z,y}) e^{\langle (z,y),(w,x)\rangle} = P_i(w,x) e^{\langle (z,y),(w,x)\rangle} \quad (z, w \in \mathrm{Sym}_n, y, x \in \mathrm{M}_{n,d})$$

Proposition 3.2 (Bernstein-Sato identity)

b-functions for cplx parameters $s = (s_1, s_2)$:

$$b_{1,0}(s) = \prod_{j=1}^{d} (s_1 + \tfrac{d+1}{2} - \tfrac{j-1}{2}) \prod_{k=1}^{n-d} (s_1 + s_2 + \tfrac{n+1}{2} - \tfrac{k-1}{2}),$$

$$b_{0,1}(s) = \prod_{j=1}^{d} (s_2 + \tfrac{d+1}{2} - \tfrac{j-1}{2})(s_2 + \tfrac{n}{2} - \tfrac{j-1}{2}) \prod_{k=1}^{n-d} (s_1 + s_2 + \tfrac{n+1}{2} - \tfrac{k-1}{2})$$

b-functions

inner product in $W_{\mathbb{C}}$: $\langle (z,y),(w,x)\rangle = \operatorname{Tr} zw + \operatorname{Tr} {}^t yx$
$P_i^*(\partial_{z,y})$: const coeff diff op defined by

$$P_i^*(\partial_{z,y}) e^{\langle (z,y),(w,x)\rangle} = P_i(w,x) e^{\langle (z,y),(w,x)\rangle} \quad (z,w \in \mathrm{Sym}_n, y,x \in \mathrm{M}_{n,d})$$

Proposition 3.2 (Bernstein-Sato identity)

b-functions for cplx parameters $s = (s_1, s_2)$:

$$b_{1,0}(s) = \prod_{j=1}^{d} (s_1 + \tfrac{d+1}{2} - \tfrac{j-1}{2}) \prod_{k=1}^{n-d} (s_1 + s_2 + \tfrac{n+1}{2} - \tfrac{k-1}{2}),$$

$$b_{0,1}(s) = \prod_{j=1}^{d} (s_2 + \tfrac{d+1}{2} - \tfrac{j-1}{2})(s_2 + \tfrac{n}{2} - \tfrac{j-1}{2}) \prod_{k=1}^{n-d} (s_1 + s_2 + \tfrac{n+1}{2} - \tfrac{k-1}{2})$$

$\Longrightarrow$ *Bernstein-Sato identity:*

$$P_1^*(\partial_{z,y})\Big(P_1(z,y)^{s_1+1} P_2(z,y)^{s_2}\Big) = b_{1,0}(s_1,s_2) P_1(z,y)^{s_1} P_2(z,y)^{s_2},$$

$$P_2^*(\partial_{z,y})\Big(P_1(z,y)^{s_1} P_2(z,y)^{s_2+1}\Big) = b_{0,1}(s_1,s_2) P_1(z,y)^{s_1} P_2(z,y)^{s_2}.$$

Enhanced positive cone

Return back to REAL WORLD $\mathbb{R}$

Enhanced positive cone

Return back to REAL WORLD $\mathbb{R}$

$$\text{Unique open orbit } /\mathbb{C} \xrightarrow{\text{breaks up}} \text{several open orbits } /\mathbb{R}$$

Enhanced positive cone

Return back to REAL WORLD $\mathbb{R}$

$$\text{Unique open orbit } /\mathbb{C} \xrightarrow{\text{breaks up}} \text{several open orbits } /\mathbb{R}$$

Among open orbits, get interested in enhanced positive cone:

$$\widetilde{\Omega} = \Omega \times \mathrm{M}^{\circ}_{n,d}(\mathbb{R})$$

$$\Omega = \mathrm{Sym}^{+}_{n}(\mathbb{R}), \quad \mathrm{M}^{\circ}_{n,d}(\mathbb{R}) \text{: full rank matrices}$$

and . . .

Enhanced positive cone

Return back to REAL WORLD $\mathbb{R}$

Unique open orbit $/\mathbb{C} \xrightarrow{\text{breaks up}}$ several open orbits $/\mathbb{R}$

Among open orbits, get interested in enhanced positive cone:

$$\widetilde{\Omega} = \Omega \times \mathrm{M}^{\circ}_{n,d}(\mathbb{R})$$

$$\Omega = \mathrm{Sym}^{+}_{n}(\mathbb{R}), \quad \mathrm{M}^{\circ}_{n,d}(\mathbb{R}): \text{ full rank matrices}$$

and . . . enhanced zeta distribution:

$$\begin{aligned} Z_{\widetilde{\Omega}}(\varphi, s) &= \int_{\widetilde{\Omega}} \varphi(z, y) P_1(z, y)^{s_1} P_2(z, y)^{s_2} dz dy \\ &= \int_{\mathrm{Sym}^{+}_{n}(\mathbb{R})} (\det z)^{s_1} \, dz \int_{\mathrm{M}_{n,d}(\mathbb{R})} \left| \det \begin{pmatrix} z & y \\ {}^{t}y & 0 \end{pmatrix} \right|^{s_2} dy \end{aligned}$$

$s = (s_1, s_2) \in \mathbb{C}^2, \quad \varphi(z, y) \in \mathscr{S}(W), \quad dz, dy$: Lebesgue measures

Meromorphic continuation

$Z_{\tilde{\Omega}}(\varphi, s)$ converges if $\operatorname{Re} s_1, \operatorname{Re} s_2 \gg 0$ $\quad \rightsquigarrow \quad \mathbb{C}^2$: meromorphic cont

Meromorphic continuation

$Z_{\tilde{\Omega}}(\varphi, s)$ converges if $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 \gg 0$ $\quad \rightsquigarrow \mathbb{C}^2$: meromorphic cont
First we need a gamma factor:

$$\Gamma_k(\alpha) = \Gamma(\alpha)\,\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{k-1}{2}) = \textstyle\prod_{j=1}^{k} \Gamma(\alpha - \tfrac{j-1}{2})$$

$$\Gamma_{\tilde{\Omega}}(s) = \Gamma_d(s_1 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{n}{2})\,\Gamma_{n-d}(s_1 + s_2 + \tfrac{n+1}{2})$$

Meromorphic continuation

$Z_{\widetilde{\Omega}}(\varphi, s)$ converges if $\operatorname{Re} s_1, \operatorname{Re} s_2 \gg 0$ $\rightsquigarrow$ $\mathbb{C}^2$: meromorphic cont
First we need a gamma factor:

$$\Gamma_k(\alpha) = \Gamma(\alpha)\,\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{k-1}{2}) = \textstyle\prod_{j=1}^{k} \Gamma(\alpha - \tfrac{j-1}{2})$$

$$\Gamma_{\widetilde{\Omega}}(s) = \Gamma_d(s_1 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{n}{2})\,\Gamma_{n-d}(s_1 + s_2 + \tfrac{n+1}{2})$$

Theorem 3.3 (Meromorphic Continuation)

The zeta integral normalized by the gamma factor

$$\frac{1}{\Gamma_{\widetilde{\Omega}}(s)} Z_{\widetilde{\Omega}}(\varphi, s)$$

is extended to an entire function in $s = (s_1, s_2) \in \mathbb{C}^2, \forall \varphi \in \mathscr{S}(W)$

Meromorphic continuation

$Z_{\widetilde{\Omega}}(\varphi, s)$ converges if $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 \gg 0$ $\rightsquigarrow$ $\mathbb{C}^2$: meromorphic cont
First we need a gamma factor:

$$\Gamma_k(\alpha) = \Gamma(\alpha)\,\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{k-1}{2}) = \textstyle\prod_{j=1}^{k} \Gamma(\alpha - \tfrac{j-1}{2})$$

$$\Gamma_{\widetilde{\Omega}}(s) = \Gamma_d(s_1 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{n}{2})\,\Gamma_{n-d}(s_1 + s_2 + \tfrac{n+1}{2})$$

Theorem 3.3 (Meromorphic Continuation)

The zeta integral normalized by the gamma factor

$$\frac{1}{\Gamma_{\widetilde{\Omega}}(s)}\, Z_{\widetilde{\Omega}}(\varphi, s)$$

is extended to an entire function in $s = (s_1, s_2) \in \mathbb{C}^2, \forall \varphi \in \mathscr{S}(W)$

$\rightsquigarrow$ $Z_{\widetilde{\Omega}}(\varphi, s)$ *extends to a meromorphic fun*
with possible poles specified by $\Gamma_{\widetilde{\Omega}}(s)$

Meromorphic continuation

$Z_{\widetilde{\Omega}}(\varphi, s)$ converges if $\operatorname{Re} s_1, \operatorname{Re} s_2 \gg 0$ $\quad\rightsquigarrow\ \mathbb{C}^2$: meromorphic cont
First we need a gamma factor:

$$\Gamma_k(\alpha) = \Gamma(\alpha)\,\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{k-1}{2}) = \textstyle\prod_{j=1}^{k} \Gamma(\alpha - \tfrac{j-1}{2})$$

$$\Gamma_{\widetilde{\Omega}}(s) = \Gamma_d(s_1 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{d+1}{2})\,\Gamma_d(s_2 + \tfrac{n}{2})\,\Gamma_{n-d}(s_1 + s_2 + \tfrac{n+1}{2})$$

Theorem 3.3 (Meromorphic Continuation)

The zeta integral normalized by the gamma factor

$$\frac{1}{\Gamma_{\widetilde{\Omega}}(s)}\, Z_{\widetilde{\Omega}}(\varphi, s)$$

is extended to an entire function in $s = (s_1, s_2) \in \mathbb{C}^2, \forall \varphi \in \mathscr{S}(W)$

$\rightsquigarrow$ $Z_{\widetilde{\Omega}}(\varphi, s)$ *extends to a meromorphic fun*

with possible poles specified by $\Gamma_{\widetilde{\Omega}}(s)$

Remark 3.4

The case for $d = 1$ is already studied by Suzuki [Suz79]

Fourier transform

Write $\tilde{z} = (z, y)$ & recall inner product on $W = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathsf{M}_{n,d}(\mathbb{R})$:

$$\langle \tilde{z}, \tilde{w} \rangle = \operatorname{Tr} zw + \operatorname{Tr} {}^t yx \qquad \text{for} \qquad \tilde{z} = (z, y),\ \tilde{w} = (w, x) \in W$$

Fourier transform

Write $\tilde{z} = (z, y)$ & recall inner product on $W = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{M}_{n,d}(\mathbb{R})$:

$$\langle \tilde{z}, \tilde{w} \rangle = \mathrm{Tr}\, zw + \mathrm{Tr}\, {}^t yx \qquad \text{for} \qquad \tilde{z} = (z, y),\ \tilde{w} = (w, x) \in W$$

Euclidean Fourier transform $\mathscr{F}\varphi = \widehat{\varphi}$ is defined as usual:

$$\widehat{\varphi}(\tilde{w}) = \int_W \varphi(\tilde{z}) e^{-2\pi i \langle \tilde{z}, \tilde{w} \rangle} d\tilde{z}, \quad \check{\psi}(\tilde{z}) = 2^{-\frac{n(n-1)}{2}} \int_W \psi(\tilde{w}) e^{2\pi i \langle \tilde{w}, \tilde{z} \rangle} d\tilde{w}$$

Fourier transform

Write $\tilde{z} = (z, y)$ & recall inner product on $W = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{M}_{n,d}(\mathbb{R})$:

$$\langle \tilde{z}, \tilde{w} \rangle = \operatorname{Tr} zw + \operatorname{Tr} {}^t y x \qquad \text{for} \qquad \tilde{z} = (z, y),\ \tilde{w} = (w, x) \in W$$

Euclidean Fourier transform $\mathscr{F}\varphi = \widehat{\varphi}$ is defined as usual:

$$\widehat{\varphi}(\tilde{w}) = \int_W \varphi(\tilde{z}) e^{-2\pi i \langle \tilde{z}, \tilde{w} \rangle} d\tilde{z}, \quad \check{\psi}(\tilde{z}) = 2^{-\frac{n(n-1)}{2}} \int_W \psi(\tilde{w}) e^{2\pi i \langle \tilde{w}, \tilde{z} \rangle} d\tilde{w}$$

Want to consider the FT of the distribution:

$$K_s^+(\tilde{z}) = \begin{cases} (\det z)^{s_1} \left| \det \begin{pmatrix} z & y \\ {}^t y & 0 \end{pmatrix} \right|^{s_2} & \tilde{z} \in \widetilde{\Omega} \\ 0 & \text{otherwise.} \end{cases}$$

Hyperfunction Ξ_s

Need one more notation: define hyperfunction

$$\Xi_s(\widetilde{w}) = P_1(+0 - 2\pi i w, x)^{s_1} P_2(+0 - 2\pi i w, x)^{s_2}$$

$$= \lim_{v \downarrow 0} \det(v - 2\pi i w)^{s_1} \left((-1)^d \det \begin{pmatrix} v - 2\pi i w & x \\ {}^t x & 0 \end{pmatrix} \right)^{s_2},$$

where $v \in \Omega = \mathrm{Sym}_n^+(\mathbb{R})$ moves to 0 in the positive cone.

Hyperfunction Ξ_s

Need one more notation: define hyperfunction

$$\begin{aligned}\Xi_s(\widetilde{w}) &= P_1(+0 - 2\pi i w, x)^{s_1} P_2(+0 - 2\pi i w, x)^{s_2} \\ &= \lim_{v \downarrow 0} \det(v - 2\pi i w)^{s_1} \left((-1)^d \det \begin{pmatrix} v - 2\pi i w & x \\ {}^t x & 0 \end{pmatrix} \right)^{s_2},\end{aligned}$$

where $v \in \Omega = \mathrm{Sym}_n^+(\mathbb{R})$ moves to 0 in the positive cone.

$\Xi_s(\widetilde{w})$ should be interpreted as

$$\int_W \Xi_s(\widetilde{w}) \varphi(\widetilde{w}) d\widetilde{w} = \lim_{v \downarrow 0} \int_W P_1(v - 2\pi i w, x)^{s_1} P_2(v - 2\pi i w, x)^{s_2} \varphi(\widetilde{w}) d\widetilde{w}$$

Hyperfunction Ξ_s

Need one more notation: define hyperfunction

$$\Xi_s(\widetilde{w}) = P_1(+0 - 2\pi i w, x)^{s_1} P_2(+0 - 2\pi i w, x)^{s_2}$$
$$= \lim_{v \downarrow 0} \det(v - 2\pi i w)^{s_1} \left((-1)^d \det \begin{pmatrix} v - 2\pi i w & x \\ {}^t x & 0 \end{pmatrix} \right)^{s_2},$$

where $v \in \Omega = \mathrm{Sym}_n^+(\mathbb{R})$ moves to 0 in the positive cone.
$\Xi_s(\widetilde{w})$ should be interpreted as

$$\int_W \Xi_s(\widetilde{w}) \varphi(\widetilde{w}) d\widetilde{w} = \lim_{v \downarrow 0} \int_W P_1(v - 2\pi i w, x)^{s_1} P_2(v - 2\pi i w, x)^{s_2} \varphi(\widetilde{w}) d\widetilde{w}$$

If $\mathrm{Re}\, s_1 \geqslant 0$ and $\mathrm{Re}\, s_2 \geqslant 0$, we can take $v \downarrow 0$ inside the integral, and get

$$\Xi_s(\widetilde{w}) = (-2\pi i)^{n s_1 + (n-d) s_2} P_1(\widetilde{w})^{s_1} P_2(\widetilde{w})^{s_2}$$

with an appropriate choice of the branch of exponents.

Hyperfunction Ξ_s

Need one more notation: define hyperfunction

$$\Xi_s(\tilde{w}) = P_1(+0 - 2\pi i w, x)^{s_1} P_2(+0 - 2\pi i w, x)^{s_2}$$

$$= \lim_{v \downarrow 0} \det(v - 2\pi i w)^{s_1} \left((-1)^d \det \begin{pmatrix} v - 2\pi i w & x \\ {}^t x & 0 \end{pmatrix} \right)^{s_2},$$

where $v \in \Omega = \mathrm{Sym}_n^+(\mathbb{R})$ moves to 0 in the positive cone.
$\Xi_s(\tilde{w})$ should be interpreted as

$$\int_W \Xi_s(\tilde{w}) \varphi(\tilde{w}) d\tilde{w} = \lim_{v \downarrow 0} \int_W P_1(v - 2\pi i w, x)^{s_1} P_2(v - 2\pi i w, x)^{s_2} \varphi(\tilde{w}) d\tilde{w}$$

If $\operatorname{Re} s_1 \geqslant 0$ and $\operatorname{Re} s_2 \geqslant 0$, we can take $v \downarrow 0$ inside the integral, and get

$$\Xi_s(\tilde{w}) = (-2\pi i)^{n s_1 + (n-d) s_2} P_1(\tilde{w})^{s_1} P_2(\tilde{w})^{s_2}$$

with an appropriate choice of the branch of exponents.
In particular, we get : $\Xi_{(0,0)}(\tilde{w}) = 1$ (constant function)

Fourier transform of K_s^+

Theorem 4.1 (Fourier transform of rel inv)

Fourier transform of K_s^+ is given by

$$\frac{1}{\Gamma_d(s_1+\frac{d+1}{2})\,\Gamma_d(s_2+\frac{n}{2})\,\Gamma_{n-d}(s_1+s_2+\frac{n+1}{2})}\widehat{K_s^+} = \frac{c(s)}{\Gamma_d(-s_2)}\,\Xi_{-(s_1+\frac{d+1}{2}),-(s_2+\frac{n}{2})}$$

where

$$c(s) = (2\pi)^{\frac{n(n-1)}{4}}\pi^{-2d(s_2+\frac{n}{4})}, \quad \Gamma_k(\alpha) = \textstyle\prod_{j=1}^{k}\Gamma(\alpha - \frac{j-1}{2})$$

Fourier transform of K_s^+

Theorem 4.1 (Fourier transform of rel inv)

Fourier transform of K_s^+ is given by

$$\frac{1}{\Gamma_d(s_1+\frac{d+1}{2})\,\Gamma_d(s_2+\frac{n}{2})\,\Gamma_{n-d}(s_1+s_2+\frac{n+1}{2})}\,\widehat{K_s^+} = \frac{c(s)}{\Gamma_d(-s_2)}\,\Xi_{-(s_1+\frac{d+1}{2}),-(s_2+\frac{n}{2})}$$

where

$$c(s) = (2\pi)^{\frac{n(n-1)}{4}}\pi^{-2d(s_2+\frac{n}{4})}, \quad \Gamma_k(\alpha) = \prod_{j=1}^{k}\Gamma(\alpha - \tfrac{j-1}{2})$$

In particular, K_s^+ has a pole at $s = -\frac{1}{2}(d+1, n)$ and there the first residue is a constant multiple of the delta distribution:

$$\delta = \frac{1}{c(s)}\frac{\Gamma_d(s_2+\frac{d+1}{2})\,\Gamma_d(-s_2)}{\Gamma_{\tilde{\Omega}}(s)} \cdot K_s^+ \Big|_{s=-\frac{1}{2}(d+1,n)}$$

Functional equation

Corollary 4.2

If $\varphi \in \mathscr{S}(W)$ is supported in the closure of the enhanced positive cone $\widetilde{\Omega}$, we get a functional equation:

$$\frac{1}{\Gamma_{\widetilde{\Omega}}(s)} Z_{\widetilde{\Omega}}(\widehat{\varphi}; s_1, s_2)$$

$$= \frac{c(s)(-2\pi i)^{-(ns_1+(n-d)s_2+\frac{n(n+1)}{2})}}{\Gamma_d(s_2 + \frac{d+1}{2})\,\Gamma_d(-s_2)} Z_{\widetilde{\Omega}}(\varphi; -(s_1 + \tfrac{d+1}{2}), -(s_2 + \tfrac{n}{2}))$$

Functional equation

Corollary 4.2

If $\varphi \in \mathscr{S}(W)$ is supported in the closure of the enhanced positive cone $\widetilde{\Omega}$, we get a functional equation:

$$\frac{1}{\Gamma_{\widetilde{\Omega}}(s)} Z_{\widetilde{\Omega}}(\widehat{\varphi}; s_1, s_2)$$

$$= \frac{c(s)(-2\pi i)^{-(ns_1+(n-d)s_2+\frac{n(n+1)}{2})}}{\Gamma_d(s_2+\frac{d+1}{2})\,\Gamma_d(-s_2)} Z_{\widetilde{\Omega}}(\varphi; -(s_1+\tfrac{d+1}{2}), -(s_2+\tfrac{n}{2}))$$

$$= \frac{c(s)(-\pi)^{-d}}{(-2\pi i)^{ns_1+(n-d)s_2+\frac{n(n+1)}{2}}} \prod_{j=1}^{d} \sin(s_2+\tfrac{d-j}{2})\pi \times$$

$$Z_{\widetilde{\Omega}}(\varphi; -(s_1+\tfrac{d+1}{2}), -(s_2+\tfrac{n}{2}))$$

Idea of Proof

$\cdots$ is straightforward

Idea of Proof

$\cdots$ is straightforward However, we need two facts:

1. Fourier transform of quadratic form (generalization of Epstein zeta integral) proved by J.L. Clerc [Cle02] (see below)

Theorem 5.1 (Clerc [Cle02, Th 2])

For a representation $E = \mathrm{M}_{n,d}(\mathbb{R})$ of $V_d = \mathrm{Sym}_d(\mathbb{R})$, Fourier transform of the power of the quadratic form $Q(y)^s = (\det {}^t yy)^s$ is given by

$$\int_E \widehat{\psi}(y)(\det {}^t yy)^s dy = \pi^{-2d(s+\frac{n}{4})} \frac{\Gamma_d(s+\frac{n}{2})}{\Gamma_d(-s)} \int_E \psi(x)(\det {}^t xx)^{-s-\frac{n}{2}} dx$$

Idea of Proof

$\cdots$ is straightforward However, we need two facts:

1. Fourier transform of quadratic form (generalization of Epstein zeta integral) proved by J.L. Clerc [Cle02] (see below)
2. Gindikin's Gamma function (cf. Faraut-Korányi [FK94])
   $\int_\Omega e^{-\operatorname{Tr} z}(\det z)^\alpha \Delta_d(z)^\beta dz = (2\pi)^{\frac{n(n-1)}{4}} \Gamma_d(\alpha+\beta+\frac{n+1}{2})\Gamma_{n-d}(\alpha+\frac{n-d+1}{2})$

Theorem 5.1 (Clerc [Cle02, Th 2])

For a representation $E = \mathrm{M}_{n,d}(\mathbb{R})$ of $V_d = \mathrm{Sym}_d(\mathbb{R})$, Fourier transform of the power of the quadratic form $Q(y)^s = (\det {}^t yy)^s$ is given by

$$\int_E \widehat{\psi}(y)(\det {}^t yy)^s dy = \pi^{-2d(s+\frac{n}{4})} \frac{\Gamma_d(s+\frac{n}{2})}{\Gamma_d(-s)} \int_E \psi(x)(\det {}^t xx)^{-s-\frac{n}{2}} dx$$

Idea of Proof

$\cdots$ is straightforward However, we need two facts:

1. Fourier transform of quadratic form (generalization of Epstein zeta integral) proved by J.L. Clerc [Cle02] (see below)
2. Gindikin's Gamma function (cf. Faraut-Korányi [FK94])
   $\int_{\Omega} e^{-\operatorname{Tr} z}(\det z)^{\alpha}\Delta_d(z)^{\beta}dz = (2\pi)^{\frac{n(n-1)}{4}}\Gamma_d(\alpha+\beta+\frac{n+1}{2})\Gamma_{n-d}(\alpha+\frac{n-d+1}{2})$

Theorem 5.1 (Clerc [Cle02, Th 2])

For a representation $E = \mathrm{M}_{n,d}(\mathbb{R})$ of $V_d = \mathrm{Sym}_d(\mathbb{R})$, Fourier transform of the power of the quadratic form $Q(y)^s = (\det {}^t yy)^s$ is given by

$$\int_E \widehat{\psi}(y)(\det {}^t yy)^s dy = \pi^{-2d(s+\frac{n}{4})}\frac{\Gamma_d(s+\frac{n}{2})}{\Gamma_d(-s)}\int_E \psi(x)(\det {}^t xx)^{-s-\frac{n}{2}}dx$$

The calculation is a fun, but it is too much involved and we omit details

Further problems

So far, we could manage the enhanced positive cone

Further problems

So far, we could manage the enhanced positive cone

Case of bilinear forms

$V = \mathrm{Sym}_n(\mathbb{R}) \rightsquigarrow (n+1)$ open orbits $\Omega(p, q)$ determined by signature

zeta distributions: $Z_{(p,q)}(\varphi, s) = \int_{\Omega(p,q)} \varphi(z) |\det z|^s dz$

Gamma factors and functional equations

Further problems

So far, we could manage the enhanced positive cone

Case of bilinear forms

$V = \mathrm{Sym}_n(\mathbb{R}) \quad \rightsquigarrow \quad (n+1)$ open orbits $\Omega(p,q)$ determined by signature

zeta distributions: $Z_{(p,q)}(\varphi, s) = \int_{\Omega(p,q)} \varphi(z) |\det z|^s dz$

Gamma factors and functional equations

$\cdots$ complete results obtained by Satake-Faraut [SF84]

Further problems

So far, we could manage the enhanced positive cone

Case of bilinear forms

$V = \mathrm{Sym}_n(\mathbb{R}) \rightsquigarrow (n+1)$ open orbits $\Omega(p, q)$ determined by signature

zeta distributions: $Z_{(p,q)}(\varphi, s) = \int_{\Omega(p,q)} \varphi(z) |\det z|^s dz$

Gamma factors and functional equations

$\cdots$ complete results obtained by Satake-Faraut [SF84]

In our case, there appear more open orbits:

Further problems

So far, we could manage the enhanced positive cone

Case of bilinear forms

$V = \mathrm{Sym}_n(\mathbb{R}) \rightsquigarrow (n+1)$ open orbits $\Omega(p,q)$ determined by signature

$$\text{zeta distributions: } Z_{(p,q)}(\varphi, s) = \int_{\Omega(p,q)} \varphi(z) |\det z|^s dz$$

Gamma factors and functional equations

$\cdots$ complete results obtained by Satake-Faraut [SF84]

In our case, there appear more open orbits:

Open orbit $\mathcal{O} \leftrightarrow (z, y) : z \in \Omega(p,q)$ and signature of $z^{-1}|_{\mathrm{Im}\, y}$

Further problems

So far, we could manage the enhanced positive cone

Case of bilinear forms

$V = \mathrm{Sym}_n(\mathbb{R}) \rightsquigarrow (n+1)$ open orbits $\Omega(p,q)$ determined by signature

zeta distributions: $Z_{(p,q)}(\varphi, s) = \int_{\Omega(p,q)} \varphi(z) |\det z|^s dz$

Gamma factors and functional equations

$\cdots$ complete results obtained by Satake-Faraut [SF84]

In our case, there appear more open orbits:

Open orbit $\mathcal{O} \leftrightarrow (z, y) : z \in \Omega(p,q)$ and signature of $z^{-1}|_{\mathrm{Im}\, y}$

Problem 6.1

Determine functional equation for arbitrary open orbits

Further and Further Problems

Another big issues are:

1. locate all poles (and zeros) (We only determined possible poles)

Further and Further Problems

Another big issues are:

1. locate all poles (and zeros) (We only determined possible poles)
2. compute residues

These are future subjects

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Here are some naïve reasons/expectations

$\cdots$ opinions/comments are welcome

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Here are some naïve reasons/expectations

$\cdots$ opinions/comments are welcome

1. It is a rare example of explicit Functional Eqs in the case of several variables (with several fundamental relative invts)

   However, $\exists$ generaltheorey for several cplx variables by Fumihiro Sato [Sat82a] [Sat83] [Sat82b]

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Here are some naïve reasons/expectations

$\cdots$ opinions/comments are welcome

1. It is a rare example of explicit Functional Eqs in the case of several variables (with several fundamental relative invts)

   However, $\exists$ generaltheorey for several cplx variables by Fumihiro Sato [Sat82a] [Sat83] [Sat82b]

2. Kernel K_s^+ was used to construct intertwiners between degenerate principal series (N-Ørsted [NOr18])

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Here are some naïve reasons/expectations

$\cdots$ opinions/comments are welcome

1. It is a rare example of explicit Functional Eqs in the case of several variables (with several fundamental relative invts)

   However, $\exists$ generaltheorey for several cplx variables by Fumihiro Sato [Sat82a] [Sat83] [Sat82b]

2. Kernel K_s^+ was used to construct intertwiners between degenerate principal series (N-Ørsted [NOr18])

   $\rightsquigarrow$ some information of images and kernels? small submodules?

Jumbles

Q Why we are interested in the enhanced zeta integrals?

Here are some naïve reasons/expectations

$\cdots$ opinions/comments are welcome

1. It is a rare example of explicit Functional Eqs in the case of several variables (with several fundamental relative invts)

   However, $\exists$ generaltheorey for several cplx variables by Fumihiro Sato [Sat82a] [Sat83] [Sat82b]

2. Kernel K_s^+ was used to construct intertwiners between degenerate principal series (N-Ørsted [NOr18])

   $\rightsquigarrow$ some information of images and kernels? small submodules?

   $\rightsquigarrow$ analytic cont of intertwiners and their residues

Jumbles continued!

3. $W = V \oplus E$ is isomorphic to a boundary orbit of the positive cone $\mathrm{Sym}^{+}_{n+d}(\mathbb{R})$ related to unitary highest weight module (Wallach set)

Jumbles continued!

3. $W = V \oplus E$ is isomorphic to a boundary orbit of the positive cone $\mathrm{Sym}^+_{n+d}(\mathbb{R})$ related to unitary highest weight module (Wallach set)

   Our results related to a further continuation of submodules restricted to a certain subgroup?

Jumbles continued!

3. $W = V \oplus E$ is isomorphic to a boundary orbit of the positive cone $\mathrm{Sym}^+_{n+d}(\mathbb{R})$ related to unitary highest weight module (Wallach set)

   Our results related to a further continuation of submodules restricted to a certain subgroup?

4. The kernel K_s^+ can be interpreted as the cplx power of a matrix coefficient of finite dimensional representation

Jumbles continued!

3. $W = V \oplus E$ is isomorphic to a boundary orbit of the positive cone $\mathrm{Sym}^{+}_{n+d}(\mathbb{R})$ related to unitary highest weight module (Wallach set)

   Our results related to a further continuation of submodules restricted to a certain subgroup?

4. The kernel K_s^{+} can be interpreted as the cplx power of a matrix coefficient of finite dimensional representation

   $\rightsquigarrow$ similar theory for Knapp-Stein kernels?
   (Barchini-Sepanski-Zierau, Ben-Saïd-Clerc-Koufany, Möllers-Oshima-Ørsted, $\cdots$)

Jumbles continued!

3. $W = V \oplus E$ is isomorphic to a boundary orbit of the positive cone $\mathrm{Sym}^+_{n+d}(\mathbb{R})$ related to unitary highest weight module (Wallach set)

   Our results related to a further continuation of submodules restricted to a certain subgroup?

4. The kernel K_s^+ can be interpreted as the cplx power of a matrix coefficient of finite dimensional representation

   $\rightsquigarrow$ similar theory for Knapp-Stein kernels?
   (Barchini-Sepanski-Zierau, Ben-Saïd-Clerc-Koufany, Möllers-Oshima-Ørsted, $\cdots$)

   $\rightsquigarrow$ constructing invariant differential operators as residues?

Thank you for your attention &

Thank you for your attention & Congratulation!! to Prof Kashiwara

- Determination of *b*-functions [SKKO81]
- Algorithm for calculating Fourier transform of zeta integrals [KM75]

References I

[Bar04] L. Barchini, *Zeta distributions and boundary values of Poisson transforms*, J. Funct. Anal. **216** (2004), no. 1, 47–70. MR 2091356

[Cle02] Jean-Louis Clerc, *Zeta distributions associated to a representation of a Jordan algebra*, Math. Z. **239** (2002), no. 2, 263–276. MR 1888224

[FK94] Jacques Faraut and Adam Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994, Oxford Science Publications. MR 1446489

[KM75] Masaki Kashiwara and Tetsuji Miwa, *Microlocal calculus and Fourier transforms of relative invariants of prehomogeneous vector spaces*, Sûrikaisekikenkyûsho Kókyûroku (1975), no. 238, 60–147, Theory of prehomogeneous vector spaces and its applications (Short Courses, Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1974). MR 0650981

[NOr18] Kyo Nishiyama and Bent Ø rsted, *Real double flag varieties for the symplectic group*, J. Funct. Anal. **274** (2018), no. 2, 573–604. MR 3724150

[Sat82a] Fumihiro Satō, *Zeta functions in several variables associated with prehomogeneous vector spaces. I. Functional equations*, Tôhoku Math. J. (2) **34** (1982), no. 3, 437–483. MR 676121

References II

[Sat82b] ———, *Zeta functions in several variables associated with prehomogeneous vector spaces. III. Eisenstein series for indefinite quadratic forms*, Ann. of Math. (2) **116** (1982), no. 1, 177–212. MR 662121

[Sat83] ———, *Zeta functions in several variables associated with prehomogeneous vector spaces. II. A convergence criterion*, Tôhoku Math. J. (2) **35** (1983), no. 1, 77–99. MR 695661

[SF84] I. Satake and J. Faraut, *The functional equation of zeta distributions associated with formally real Jordan algebras*, Tohoku Math. J. (2) **36** (1984), no. 3, 469–482. MR 756029

[SKKO81] M. Sato, M. Kashiwara, T. Kimura, and T. Ōshima, *Microlocal analysis of prehomogeneous vector spaces*, Invent. Math. **62** (1980/81), no. 1, 117–179. MR 595585

[SS74] Mikio Sato and Takuro Shintani, *On zeta functions associated with prehomogeneous vector spaces*, Ann. of Math. (2) **100** (1974), 131–170. MR 0344230

[Suz79] Toshiaki Suzuki, *On zeta functions associated with quadratic forms of variable coefficients*, Nagoya Math. J. **73** (1979), 117–147. MR 524011