Dual pairs and classical invariant theory

Representation theory and recent advances

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Seminar Talk at Department of Mathematics, Bangalore University

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We will explain only a small part of the relationship
between the theory of dual pairs and the classical invariant theory

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Definition (dual pair)

(G, G') : reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$ if

1. G, G' are reductive subgroups in $\mathrm{Sp}(W_{\mathbb{R}})$
2. mutually commutant to each other in $\mathrm{Sp}(W_{\mathbb{R}})$

$$G' = \{h \in \mathbb{G} \mid gh = hg \quad (\forall g \in G)\}$$
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$\mathbb{K} \subset \mathbb{G}$, $K \subset G$, $K' \subset G'$: maximal compact subgroups
We can assume: $K \cdot K' \subset \mathbb{K}$

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U : vector space $/\mathbb{F}$ with ι-anti-Hermitian form $(\,,\,)_U$
$\Longrightarrow$ $W = V \otimes_{\mathbb{F}} U$ with ι-anti-Hermitian form $(\,,\,)_W = (\,,\,)_V \otimes (\,,\,)_U$
$\Longrightarrow$ $\langle x, y \rangle = \mathrm{Im}(x, y)_W$: symplectic form on $W_{\mathbb{R}} := W/\mathbb{R}$

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Put $G := \mathrm{U}^{+}(V) \otimes 1 \hookrightarrow \mathbb{G}$: isometry group wrt $(\,,\,)_V$
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Lemma

(G, G') is a reductive dual pair inside $\mathbb{G}$

Table of type I dual pairs

field	(G, G')	stable range
$\mathbb{R}$:	$(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$	$2n \le \min(p,q)$
$\mathbb{C}$:	$(\mathrm{U}(p,q), \mathrm{U}(m,n))$	$m + n \le \min(p,q)$
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$\mathrm{O}(p,q)$: orthogonal group preserving indefinite quadraric form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

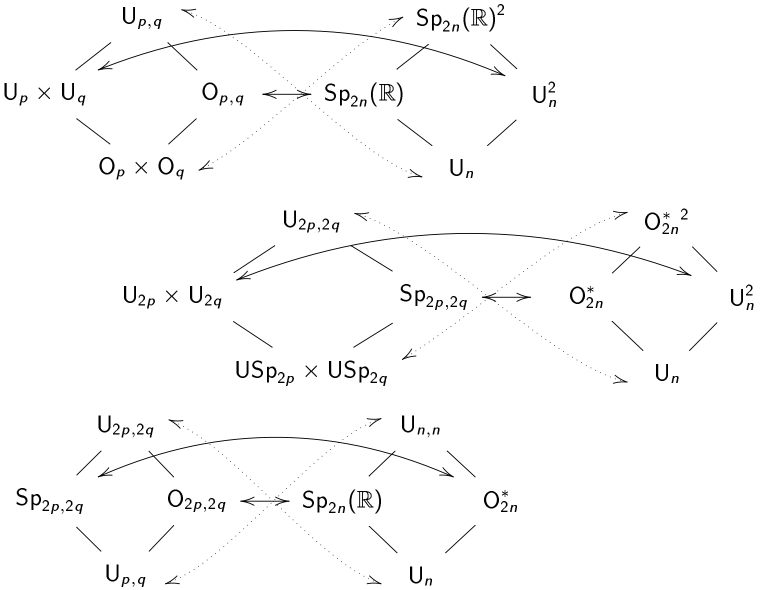
$\mathrm{U}(p,q)$: indefinite unitary group preserving Hermitian form

$$x_1\overline{y_1} + \cdots + x_p\overline{y_p} - x_{p+1}\overline{y_{p+1}} - \cdots - x_{p+q}\overline{y_{p+q}}$$

$\mathrm{Sp}(2n,\mathbb{R})$: symplectic group preserving symplectic form

$p = 0$ or $q = 0 \Rightarrow$ dual pair of compact type

See-saw pair & diamond pair



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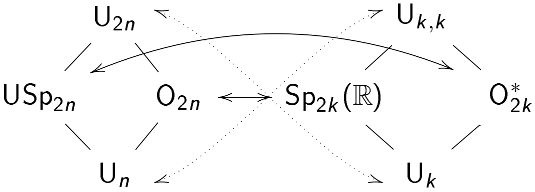

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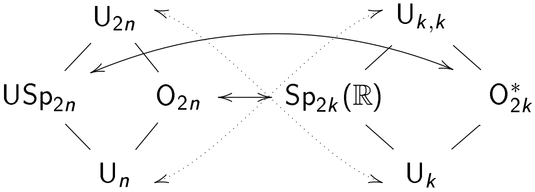

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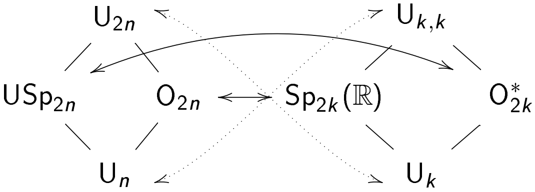

$(G, G') \subset Sp(W_{\mathbb{R}})$: dual pair with G compact (one of the above)

$W_{\mathbb{C}} = W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = X \oplus Y$: $\exists$ polar decomposition $/\mathbb{C}$ s.t.

1. X & Y : max totally isotropic
2. $K' \subset G'$: max compact subgrp $\overset{\mathbb{C}\text{-fy}}{\Longrightarrow}$ $K'_{\mathbb{C}}$
   $G_{\mathbb{C}} \times K'_{\mathbb{C}} \curvearrowright X$: holomorphic action

$G \curvearrowright X \implies G$ acts on functions on X

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$\mathbb{C}[X]^G = \{f(x) \in \mathbb{C}[X] \mid \pi(g)f(x) = f(x)\}$: graded algebra

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$G = O_n(\mathbb{R}) \curvearrowright X = \mathbb{C}^n \qquad \mathbb{C}[X]^G = \mathbb{C}[\xi]$

$\xi = x_1^2 + \cdots + x_n^2 \leftrightarrow$ Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$

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$$\mathcal{H}(G) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \Delta f = 0\} = \bigoplus_{\ell=0}^{\infty} V(\ell)$$

$V(\ell) =$ (spherical harmonics of deg ℓ)

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$$\mathbb{C}[X] = \mathcal{H}(G) \otimes \mathbb{C}[\xi] = \bigoplus_{k=0}^{\infty} \mathcal{H}(G)\,\xi^k$$
$$\mathcal{H}(G) = \bigoplus_{\ell=0}^{\infty} V(\ell)$$

$V(\ell)\big|_{S^{n-1}}$ = polynomials of deg ℓ restricted to the sphere S^{n-1}

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$(G, G') = (\mathrm{O}_m(\mathbb{R}), \mathrm{Sp}_{2k}(\mathbb{R}))$ in $\mathrm{Sp}_{2mk}(\mathbb{R})$; $G_{\mathbb{C}} = \mathrm{O}_m(\mathbb{C})$, $K'_{\mathbb{C}} = \mathrm{GL}_k(\mathbb{C})$

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where

$\mathcal{P}_k = \{\lambda = (\lambda_1, \ldots, \lambda_k) \mid \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}$

σ_λ: irred finite dim rep of O_m with ht wt λ

ρ_λ: irred finite dim rep of GL_k

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Remark

Frobenius reciprocity tells us that $\dim(\sigma_\lambda)^{\mathrm{O}_{m-k}} = \dim \rho_\lambda$

Complete description of $\mathbb{C}[\mathrm{M}_{m,k}]$ as a $\mathrm{O}_m \times \mathrm{GL}_k$-module

$X = \mathrm{M}_{m,k}$ Assume $m > 2k$

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Tracing the effect of degree, we get

Theorem

$\mathbb{C}[\mathrm{M}_{m,k}]_d \simeq \displaystyle\bigoplus_{\lambda,\mu \in \mathcal{P}_k} \sigma_\lambda \otimes (\rho_\lambda \otimes \rho_{2\mu})$ *where* $\lambda, \mu \in \mathcal{P}_k$ *moves under the condition* $|\lambda| + 2|\mu| = d$

Example

$(G, G') = (\mathrm{U}_n, \mathrm{U}_{k,k})$ in $\mathrm{Sp}_{4nk}(\mathbb{R})$;
$G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C}),\ K'_{\mathbb{C}} = \mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C})$

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$G_{\mathbb{C}} \curvearrowright X = \mathrm{M}_{n,2k}(\mathbb{C}) \curvearrowleft K'_{\mathbb{C}}$
where $X = M_{2n,k}(\mathbb{C}) = (\mathbb{C}^n)^* \otimes \mathbb{C}^k \oplus \mathbb{C}^n \otimes (\mathbb{C}^k)^*$

Assume $n \geq 2k$ (stable range condition) then

$$\mathbb{C}[X] \simeq \mathcal{H}(\mathrm{GL}_n) \otimes \mathbb{C}[X]^{\mathrm{GL}_n}$$

invariants: $\mathbb{C}[X]^{\mathrm{GL}_n} \simeq \mathbb{C}[\mathrm{M}_k]$

harmonics: $\mathcal{H}(\mathrm{GL}_n) \simeq \bigoplus_{\alpha,\beta \in \mathcal{P}_k} \rho^{(n)}_{\alpha \odot \beta} \otimes (\rho^{(k)}_\alpha \otimes \rho^{(k)}_\beta)$
$(\mathrm{GL}_n \times (\mathrm{GL}_k \times \mathrm{GL}_k)\text{-mod})$

$\alpha \odot \beta = (\alpha_1, \dots, \alpha_k, 0, \dots 0, -\beta_k, -\beta_{k-1}, \dots, -\beta_k)$
$\rho^{(k)}_\lambda$: irred finite dim rep of GL_k

Example

$(G, G') = (USp_{2n}, O^*_{2k})$ in $Sp_{4nk}(\mathbb{R})$; $G_{\mathbb{C}} = Sp_{2n}(\mathbb{C}),\ K'_{\mathbb{C}} = GL_k(\mathbb{C})$

$W_{\mathbb{C}} = M_{2n,2k}(\mathbb{C}) = M_{2n,k}(\mathbb{C}) \oplus M_{2n,k}(\mathbb{C}) := X \oplus Y$: polar decomp
$G_{\mathbb{C}} \curvearrowright X = M_{2n,k}(\mathbb{C}) \curvearrowleft K'_{\mathbb{C}}$ by multiplication

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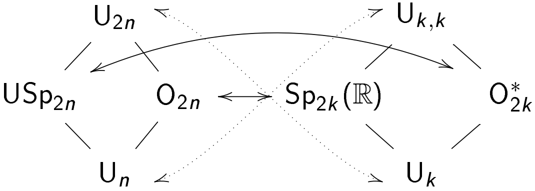
where
τ_λ: irred finite dim rep of Sp_{2n} with ht wt λ
ρ_λ: irred finite dim rep of GL_k

Intersection of harmonics

Recall the diamond pair of cpt type:



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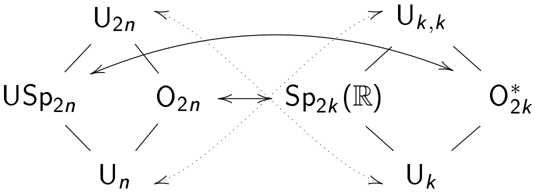

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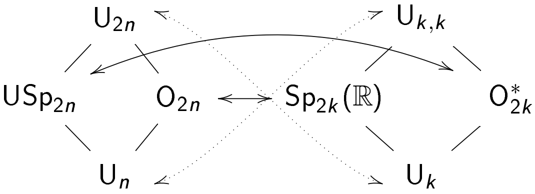

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$\Longrightarrow$ we can take intersection

$$\mathcal{H}(U_n) = \mathcal{H}(O_{2n}) \cap \mathcal{H}(USp_{2n})$$

Theorem

Put $c^{\lambda}_{\alpha,\beta}$*: Littlewood-Richardson coeffcient*

$$\mathcal{H}(\mathrm{GL}_n) = \mathcal{H}(\mathrm{O}_{2n}) \cap \mathcal{H}(\mathrm{Sp}_{2n}) = \bigoplus\nolimits_{\alpha,\beta,\lambda\in\mathcal{P}_k} c^{\lambda}_{\alpha,\beta}\, \rho^{(n)}_{\alpha\odot\beta} \otimes \rho^{(k)}_{\lambda}$$

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$$(\sigma_\lambda \otimes \rho^{(k)}_{\lambda}) \cap (\tau_\lambda \otimes \rho^{(k)}_{\lambda}) = \bigoplus_{|\lambda|=|\alpha|+|\beta|} c^{\lambda}_{\alpha,\beta}\, \rho^{(n)}_{\alpha\odot\beta} \otimes \rho^{(k)}_{\lambda} \quad (\alpha, \beta \in \mathcal{P}_k)$$

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Fix $v_\lambda \in \rho^{(2n)}_\lambda$: GL_{2n}-highest weight vector

$\implies \sigma_\lambda \in \mathrm{Irr}(\mathrm{O}_{2n}), \tau_\lambda \in \mathrm{Irr}(\mathrm{Sp}_{2n})$: subrep of $\rho^{(2n)}_\lambda$ generated by v_λ

$$\rho^{(2n)}_\lambda \supset \sigma_\lambda \cap \tau_\lambda = \bigoplus_{|\lambda|=|\alpha|+|\beta|} c^{\lambda}_{\alpha,\beta}\, \rho^{(n)}_{\alpha\odot\beta}$$

(n should be sufficiently large)

Weil representation

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$\widetilde{\mathbb{G}} := \mathrm{Mp}(W_{\mathbb{R}})$: metaplectic group

$\zeta : \widetilde{\mathbb{G}} = Mp(W_{\mathbb{R}}) \to \mathbb{G} = \mathrm{Sp}(W_{\mathbb{R}})$: $\exists 1$ non-trivial double cover

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3. unitary highest weight modules with singular inf char

Realization of Ω on $\mathbb{C}[X]$

$\Omega = L^2(X_{\mathbb{R}})_{\widetilde{\mathbb{K}}}$: $\widetilde{\mathbb{K}}$-finite vectors in ω (Harish-Chandra module of ω)

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Action of $\widetilde{\mathbb{K}}$

$$\Omega(k)f(z) = \chi(k)\, f(\zeta(k)^{-1}z) \quad (k \in \widetilde{\mathbb{K}}, f(z) \in \mathbb{C}[X], z \in X)$$

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Infinitesimal action of $\widetilde{\mathbb{G}}/\widetilde{\mathbb{K}}$

$\mathrm{Lie}(\widetilde{\mathbb{G}}) = \mathfrak{K} \oplus \mathfrak{P}$: $\mathbb{C}$-fied Cartan decomp $\quad \mathfrak{P} = T_{e\widetilde{K}}(\widetilde{\mathbb{G}}/\widetilde{\mathbb{K}})$

$\{X_{\pm(\varepsilon_i+\varepsilon_j)} \mid 1 \leq i, j \leq N\}$: root vectors in $\mathfrak{P}$

$$\Omega(X_{\varepsilon_i+\varepsilon_j})f(z) = \frac{\partial^2}{\partial z_i \partial z_j} f(z) \qquad : \text{positive noncpt roots}$$

$$\Omega(X_{-\varepsilon_i-\varepsilon_j})f(z) = z_i z_j f(z) \qquad : \text{negative noncpt roots}$$

(G, G'): dual pair of compact type

See-saw pair
$$\begin{array}{ccc} M & & G' \\ | & \nwarrow\!\!\!\!\nearrow & | \\ & \swarrow\!\!\!\!\searrow & \\ G & & K' \end{array} \qquad \text{Ex.} \quad \begin{array}{ccc} U_{2n} & & Sp_{2k}(\mathbb{R}) \\ | & \nwarrow\!\!\!\!\nearrow & | \\ & \swarrow\!\!\!\!\searrow & \\ O_{2n} & & U_k \end{array}$$

G'/K': Hermitian symm space

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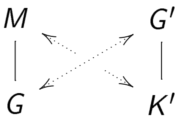
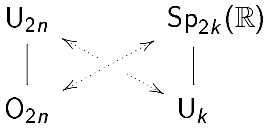
See-saw pair 

Ex. 

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Facts

1. $\Omega(U(\mathfrak{p}'_+)) = \mathbb{C}[X]^G$: multiplication by invariants
2. $\Omega(U(\mathfrak{p}'_-)) = \partial(\mathbb{C}[X]^G)$: differential by invariants
3. $\mathcal{H}(G) = \{f \in \mathbb{C}[X] \mid \Omega(\mathfrak{p}'_-)f = 0\}$: harmonics

Recall $\mathcal{H}(G) = \bigoplus_{\sigma \in \mathrm{Irr}(G)} \sigma \otimes \rho$: mult free decomp (σ determines ρ)

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as $G \times (\mathfrak{g}', \widetilde{K'})$-module

For any $v \neq 0 \in \sigma$,

$L_\sigma \simeq \Omega(U(\mathfrak{p}'_+))(v \otimes \rho)$: unitary highest weight mod
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Conclusion:

> Harmonics = Lowest K-types of unitary ht wt module

Capelli identity for symmetric pair

joint work with Soo Teck Lee and Akihito Wachi [math.RT/0510033]

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Take K'-invariants : $\left(S(\mathfrak{p}'_+) \otimes S(\mathfrak{p}'_-)\right)^{K'} \to \Omega\left(U(\mathfrak{g}')^{K'}\right)$

$\therefore \forall X \in \left(S_+(\mathfrak{p}'_+) \otimes S_+(\mathfrak{p}'_-)\right)^{K'} \implies \exists Z \in U(\mathfrak{m})^G$ s.t. $\Omega(X) = \Omega(Z)$

Proof.

- (M, K') is dual pair & X is K'-invariant $\implies \Omega(X) \in \Omega(U(\mathfrak{m}))$
- (G, G') is dual pair & $X \in U(\mathfrak{g}')$ $\implies \Omega(X) \in \Omega(U(\mathfrak{m})^G)$

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If G is suff large, Z is unique

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□

If G is suff large, Z is unique

Definition

- $Z = C_X \in U(\mathfrak{m})^G$: Capelli element for $X \in U(\mathfrak{g}')^{K'}$
- $\Omega(X) = \Omega(Z)$: Capelli identity for symmetric pairs

Example

$(M, G) = (\mathrm{U}_{2n}, \mathrm{O}_{2n}),\ (G', K') = (\mathrm{Sp}_{2k}(\mathbb{R}), \mathrm{U}_k)$

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$$D = (D_{i,j})_{1 \le i,j \le k} \quad D_{i,j} = \sqrt{-1} \sum_{r=1}^{2n} \partial_{ri} \partial_{rj}$$

$$E = (E_{r,s})_{1 \le r,s \le 2n} \quad E_{r,s} = \sum_{i=1}^{k} z_{ri} \partial_{si}$$

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Capelli identity:

$$\sum_{I,J \in \mathcal{I}_d^k} \det X_{I,J} \det D_{I,J} = \sum_{R,S \in \mathcal{I}_d^{2n}} \det\left(E_{R(i),S(j)} + (d - j - 1)\delta_{R(i)S(j)}\right)_{i,j} \times \det\left(E_{R(i),S(j)} + (d - j)\delta_{R(i)S(j)}\right)_{i,j}$$

$\mathcal{I}_d^m = \{I \subset [m] \mid \# I = d\}, [m] = \{1, 2, \ldots, m\}$

Example

$(M, G) = (\mathrm{U}_{2n}, \mathrm{USp}_{2n})$, $(G', K') = (\mathrm{O}^*_{2k}, \mathrm{U}_k)$

$$G_{ij} = \sqrt{-1} \sum_{s=1}^{n} (x_{si} x_{\bar{s}j} - x_{\bar{s}i} x_{sj}) \qquad (\bar{s} := s + n)$$

$$F_{ji} = \sqrt{-1} \sum_{s=1}^{n} (\partial_{si} \partial_{\bar{s}j} - \partial_{\bar{s}i} \partial_{sj})$$

$$E_{st} = \sum_{i=1}^{k} x_{si} \partial_{ti} + \frac{k}{2} \delta_{st}$$

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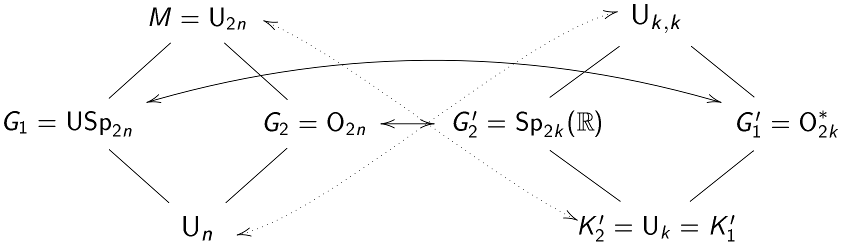
Capelli identity: $d = 1, 2, \ldots, r; \quad r = \lfloor k/2 \rfloor$

$$\sum_{I \in \mathcal{I}_{2d}^k} \mathrm{Pf}\, G_{II} \cdot \mathrm{Pf}\, F_{II} = \sum_{S_0, T_0 \in \mathcal{I}_d^n} \det(E_{s_a, t_b} + (2d - b - k/2)\delta_{s_a, t_b})_{1 \le a,b \le 2d}$$

where for $S_0 = \{s_1, s_2, \ldots, s_d\} \in \mathcal{I}_d^n$, we put

$$S := \{s_1, s_2, \ldots, s_d; \overline{s_1}, \overline{s_2}, \ldots, \overline{s_d}\} \in \mathcal{I}_{2d}^{2n}$$

Recall diamond pair of compact type



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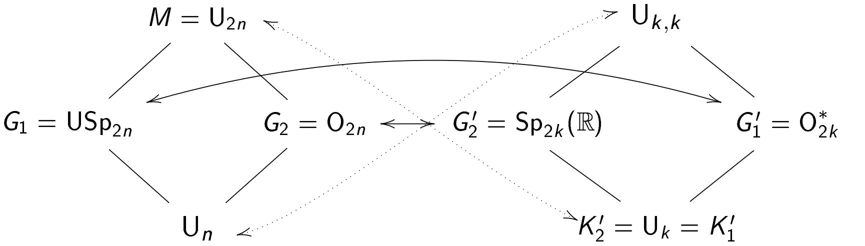

$\mathcal{H}(G_i)$ $(i = 1, 2)$ is killed by $\Omega(X_i)$ $\quad (\forall X_i \in \left(S_+(\mathfrak{p}'_{\mathfrak{i}+}) \otimes S_+(\mathfrak{p}'_{\mathfrak{i}-})\right)^{K_i'})$

$\Longrightarrow$ killed by $\Omega(C_{X_i})$ $\quad (C_{X_i} \in U(\mathfrak{m})^{G_i})$

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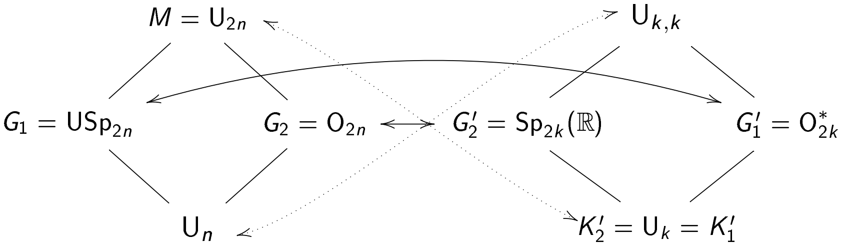

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Theorem

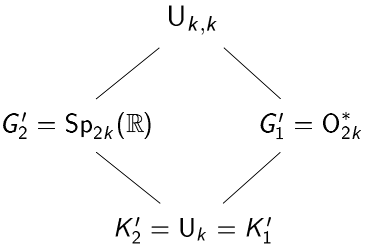
Capelli element $\Omega(C_{X_i})$ $(i = 1, 2)$ *kills the intersection* $\sigma_\lambda \cap \tau_\lambda$.

Remark

Note that $\Omega(C_X) \doteqdot \rho_\lambda^{(2n)}(C_X)$. They only differ by a character.

Branching of unitary ht wt module

Branching for the subgroups



L_σ : UHW of $U_{k,k} \longleftrightarrow \sigma \otimes \rho \in \mathcal{H}(GL_n)$: harmonics
$\sigma \in \mathrm{Irr}(U_n)$, Lowest K-type $= \rho(\sigma) \in \mathrm{Irr}(U_k \times U_k)$

$$\mathcal{H}(Sp_{2n}) \cap \mathcal{H}(O_{2n}) \subset \mathcal{H}(GL_n) \longleftrightarrow (\text{UHW of } Sp_{2k}(\mathbb{R})) \cap (\text{UHW of } O_{2k}^*)$$

They are killed by Capelli elements!

Problem

- *Describe the intersection* (*UHW of* $\mathrm{Sp}_{2k}(\mathbb{R})$) $\cap$ (*UHW of* O^*_{2k})
  If $n =$ (*size of compact member*) $\geq k$,
  the intersection is fin dim, $=$ *intersection of harmonics*

Problem

- *Describe the intersection* $(UHW \text{ of } \mathrm{Sp}_{2k}(\mathbb{R})) \cap (UHW \text{ of } \mathrm{O}^*_{2k})$
  If $n =$ (*size of compact member*) $\geq k$,
  the intersection is fin dim, $=$ *intersection of harmonics*
- *Characterize the intersection by differential operators*
  Need more than Capelli elements?

Thank you for your attention!