Steinberg variety and moment maps over multiple flag varieties I

Kyo Nishiyama

Dept. Mathematics, Aoyama Gakuin Univ.

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Plan of talk

1. Conormal variety
   Review the properties of conormal varieties
2. Steinberg variety for $G/B \times G/B$
   Introduce classical Steinberg variety
3. Springer representation in type A
   Introduce standard tableaux and Springer representations for type A
4. Robinson-Schensted correspondence
   How RS-correspondence arises in Steinberg theory
5. Steinberg theory for KGB
   Generalize the Steinberg theory to a symmetric pair (G, K)
6. KGB decomposition for type A
   Formula of the K orbits for type A

Conormal variety

G : algebraic group / $\mathbb{C}$ $\qquad \mathfrak{g} = \mathrm{Lie}\,(G)$

X : smooth variety $\curvearrowleft G$

T^*X : cotangent bdle (symplectic) $\curvearrowleft G$ by Hamiltonian action

$$\Longrightarrow \exists \text{ moment map } \mu : \begin{array}{ccc} T^*X & \longrightarrow & \mathfrak{g}^* \\ \Cup & & \Cup \\ (x, \xi) & \longmapsto & (z \mapsto \xi(z_x)) \end{array}$$

z_x : vector field at $x \in X$ generated by $z \in \mathfrak{g}$

Definition

$S_X := \mu^{-1}(0) \subset T^*X$: conormal variety

$G \backslash X \ni \mathbb{O}$: G-orbit $\rightsquigarrow T^*_{\mathbb{O}}X$: conormal bdle $\quad (\mathbb{N}^*_{\mathbb{O} \subset X}$ in Joe's notation)

$G\backslash X \ni \mathbb{O}$: G-orbit $\rightsquigarrow$ $T^*_{\mathbb{O}}X$: conormal bdle

Lemma

$S_X = \bigsqcup_{\mathbb{O}\in G\backslash X} T^*_{\mathbb{O}}X$ *(hence the name of conormal variety)*

Proof.

$$(x,\xi) \in \mu^{-1}(0) \iff \mu(x,\xi)(z) = \xi(z_x) = 0 \quad (\forall z \in \mathfrak{g})$$
$$\iff \xi \in (T^*_{\mathbb{O}}X)_x \quad (\mathbb{O} := G\cdot x)$$

□

Corollary

Assume $\#G\backslash X < \infty$.

1. S_X *is equi-dimensional of* $\dim X$ *and*
2. $S_X = \bigcup_{\mathbb{O}\in G\backslash X} \overline{T^*_{\mathbb{O}}X}$ *gives irred decomposition as an alg variety*

($\because$ $T^*_{\mathbb{O}}X$: irreducible and $\dim T^*_{\mathbb{O}}X = \dim X$)

Steinberg variety

G : reductive $\supset B$: Borel subgrp $\supset T$: max torus
$G/B = \{gB \mid g \in G\} = \{\forall$ Borel subgrps$\} \simeq \{\forall$ Borel subalgbras$\}$
$X := G/B \times G/B \curvearrowleft G$: diag action

Lemma

$G\backslash X \simeq B\backslash G/B = \bigsqcup_{w \in W} BwB$: *Bruhat decomp*
where $W := N_G(T)/T$: *Weyl group*

($\because$ $G \cdot (hB, kB) \longmapsto Bh^{-1}kB$ gives a bijection)

Example

$G = \mathrm{GL}_n(\mathbb{C}) \supset B =$ (upper triangular matrices) $\supset T =$ (diag matrices)
$W =$ (permutation matrices) $\rightsquigarrow$ Bruhat decomp $\fallingdotseq$ LPU decomp

LPU = (Lower triang) · (Permutation) · (Upper triang)

$X := G/B \times G/B \curvearrowleft G$: diag action

$\# G \backslash X = \# B \backslash G/B = \# W < \infty \rightsquigarrow$ can apply Corollary

$G/B \simeq \{\mathfrak{b}' : \text{Borel subalg}\}$: flag variety

$$T^*(G/B) \simeq \{(\mathfrak{b}', \xi) \mid \xi \in (\mathfrak{g}/\mathfrak{b}')^\perp \simeq \mathfrak{u}'\}$$

$\mathfrak{b}' \supset \mathfrak{u}'$: nilpotent radical $\rightsquigarrow$ $\mathfrak{u}' \simeq (\mathfrak{g}/\mathfrak{b}')^\perp$ via Killing form

$$\simeq \{(\mathfrak{b}', u) \mid u \in \mathfrak{b}' : \text{nilpotent}\}$$

Two moment maps

$$\mu_{G/B} : \begin{array}{ccc} T^*(G/B) & \longrightarrow & \mathfrak{g} \simeq \mathfrak{g}^* \\ \Cup & & \Cup \\ (\mathfrak{b}', u) & \longmapsto & u \end{array}$$

$$\mu_X : \begin{array}{ccc} T^*X = T^*(G/B) \times T^*(G/B) & \longrightarrow & \mathfrak{g} \\ \Cup & & \Cup \\ ((\mathfrak{b}', u), (\mathfrak{b}'', v)) & \longmapsto & u + v \end{array}$$

$$\mu_X^{-1}(0) = \{((\mathfrak{b}', u), (\mathfrak{b}'', v)) \mid u \in \mathfrak{u}', v \in \mathfrak{u}'', \ u + v = 0\} : \text{conormal var}$$

$\mu_X^{-1}(0) = \{((\mathfrak{b}', u), (\mathfrak{b}'', v)) \mid u \in \mathfrak{u}', v \in \mathfrak{u}'', \; u + v = 0\}$: conormal var

Definition (Steinberg variety)

$S_X := \{(\mathfrak{b}', \mathfrak{b}'', u) \mid u \in \mathfrak{b}' \cap \mathfrak{b}'' : \text{nilpotent}\}$

- $G \backslash X = \{\mathbb{O}_w := G \cdot (\mathfrak{b}, w\mathfrak{b}) \;\; (w \in W)\}$ by Bruhat decomp
- $S_X = \bigcup_{w \in W} \overline{T^*_{\mathbb{O}_w} X}$: $\#W$-irred components
- $\mu_{G/B}(\overline{T^*_{\mathbb{O}_w} X}) = G \cdot (\mathfrak{u} \cap (w \cdot \mathfrak{u})) = \overline{\mathcal{O}}$ where $\mathcal{O}$: nilpotent G-orbit in $\mathfrak{g}$

  $\rightsquigarrow \;\; \Phi_X : G \backslash X \simeq W \to \mathcal{N}(\mathfrak{g})/G$: orbit map

Fibers of moment map:

$\mathcal{O} \in \mathcal{N}(\mathfrak{g})/G$: nilpotent orbit, $u \in \mathcal{O}$

1. $\mathscr{B}_u := \mu_{G/B}^{-1}(u)$: Spaltenstein variety (= Springer fiber)
2. $W(\mathcal{O}) := \Phi_X^{-1}(\mathcal{O})$: Steinberg cell (cf Robinson-Schensted corr)

Why Spaltenstein & Springer?

Why Robinson-Schensted?

Explain this in the case of type A

type A

$G = \mathrm{GL}_n(\mathbb{C}) \supset B = (\text{upper triangular}) \supset T = (\text{diagonals})$
$W = S_n$: symmetric group $\qquad \lambda \in \mathscr{P}(n)$: partitions of n

$\mathrm{STab}(\lambda) = \{$standard tableaux of shape $\lambda\}$

→ increasing

↓ increasing

1	3	6
2	5	7
4		
8		

$\mathrm{STab}(\lambda) \overset{\text{Specht module}}{\rightsquigarrow}$ basis of $\sigma_\lambda \in \mathrm{Irr}(W) = $ (irred reps of W)

$$\begin{array}{ccccc} \mathcal{N}(\mathfrak{g})/G & \xleftrightarrow{\text{Jordan NF}} & \mathscr{P}(n) & \xleftrightarrow{\text{Specht}} & \mathrm{Irr}(W) \\ \Cup & & \Cup & & \Cup \\ \mathcal{O}_\lambda & \longleftrightarrow & \lambda & \longleftrightarrow & \sigma_\lambda \end{array}$$

Theorem (Spaltenstein variety)

For $u \in \mathcal{O}_\lambda$

1. $\mathscr{B}_u := \mu_{G/B}^{-1}(u)$ *is equi-dimensional of dim* $= \frac{1}{2}(n^2 - n) - \dim \mathcal{O}_\lambda$
2. $\#\mathrm{Irr}(\mathscr{B}_u) = f_\lambda = \#\mathrm{STab}(\lambda)$

How to associate tableaux to irreducible components of $\mathscr{B}_u$?

Correspondence of Borels and flags

$G/B \ni \mathfrak{b}' \longleftrightarrow$ (flag of subspaces in $\mathbb{C}^n$)
where flag $\mathscr{F} := (F_k)_{0 \leq k \leq n}$ is a sequence of subspaces:

$$F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n \quad \text{s.t. } \dim F_k = k$$

$$\mathfrak{b}' = \mathrm{Stab}_{\mathfrak{g}}((F_k)) = \{z \in \mathfrak{g} \mid zF_k \subset F_k\}$$

$u \in \mathfrak{b}' \implies uF_k \subset F_k$: nilpotent endomorphism of F_k
Thus we get $\mathscr{B}_u \simeq \{\mathfrak{b}' : \text{Borel subalg} \mid u \in \mathfrak{b}'\}$

$$\simeq \{\mathscr{F} = (F_k)_{0 \leq k \leq n} : \text{flag} \mid u \cdot F_k \subset F_k\}$$

How to describe irred components $\mathrm{Irr}(\mathscr{B}_u)$?

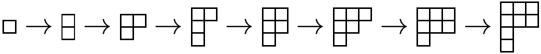
$\mathscr{F} = (F_k)_{0 \le k \le n} \in \mathscr{B}_u$

$\rightsquigarrow$ $u|_{F_k}$: nilpotent endomorphism in F_k

$\rightsquigarrow$ $u|_{F_k}$ determines partition $\lambda^{(k)} \in \mathscr{P}(k)$

So we get:

$\lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(n-1)} \subset \lambda^{(n)}$: developing sequence of partitions



Put $\#$ of stages in each box in the development, we obtain

1	3	6
2	5	7
4		
8		

: standard tableau

Theorem (Springer representation)

For $u \in \mathcal{O}_\lambda$: nilpotent element

1. *Standard tableaux* $\mathrm{Stab}(\lambda)$ *determines* $\mathrm{Irr}(\mathscr{B}_u)$ *completely* $(u \in \mathcal{O}_\lambda)$
2. *Top Borel-Moore cohomology* $H^{\mathrm{top}}_{BM}(\mathscr{B}_u)$ *carries irred repr* $\sigma_\lambda \in \mathrm{Irr}(W) \leftrightarrow$ *basis parametrized by* $\mathrm{Stab}(\lambda)$

What can we say about Steinberg variety S_X?

Consider product of moment maps:

$\mu^2_{G/B} := \mu_{G/B} \times \mu_{G/B} : T^*X = T^*(G/B) \times T^*(G/B) \to \mathfrak{g} \times \mathfrak{g}$

$\rightsquigarrow \ \mu^2_{G/B} : S_X \to \mathcal{N}(\mathfrak{g})$: G-equiv map

$$(\because \ \mu^2_{G/B}(S_X) = \{(u, -u) \mid u \in \mathcal{N}(\mathfrak{g})\} \simeq \mathcal{N}(\mathfrak{g}))$$

Fiber : $(\mu^2_{G/B})^{-1}(u) = \mathscr{B}_u \times \mathscr{B}_u \quad (\because \mathscr{B}_{-u} = \mathscr{B}_u)$

Recall the G-orbit $\mathbb{O}_w \in G\backslash X \simeq B\backslash G/B$ &
Steinberg cell $W(\mathcal{O}_\lambda) = \Phi_X^{-1}(\mathcal{O}_\lambda)$ $(\mathcal{O}_\lambda \in \mathcal{N}(\mathfrak{g})/G)$

Lemma

$\mathbb{O}_w \in W(\mathcal{O}_\lambda) \iff \overline{(\mathscr{B}_u \times \mathscr{B}_u) \cap T^*_{\mathbb{O}_w} X}$ *is an irred comp of* $\mathscr{B}_u \times \mathscr{B}_u$

This shows the correspondence:

$$W(\mathcal{O}_\lambda) \longleftrightarrow \mathrm{Irr}(\mathscr{B}_u \times \mathscr{B}_u) \longleftrightarrow \{(T_1, T_2) \mid T_i \in \mathrm{STab}(\lambda)\}$$

Robinson-Schensted correspondence

$W = \bigsqcup_{\mathcal{O} \in \mathcal{N}(\mathfrak{g})/G} W(\mathcal{O})$: cell decomposition

Thus we get

$$w \in W \longleftrightarrow \mathbb{O}_w \in G \backslash X \longleftrightarrow C_w \in \mathrm{Irr}(\mathscr{B}_u \times \mathscr{B}_u)$$
$$\longleftrightarrow (T_1, T_2) \in \mathrm{STab}(\lambda)^2$$

(λ is determined by $\Phi_X(\mathbb{O}_w) = \mathcal{O}_\lambda$)

Theorem (Steinberg)

The above correspondence : $W = S_n \ni w \longleftrightarrow (T_1, T_2) \in \mathrm{STab}(\lambda)^2$ *gives the Robinson-Schensted corr for type A*

Generalization to symmetric pairs

G : reductive algebraic group $/\mathbb{C}$
(G, K) : symmetric pair i.e., $\exists\theta$: involution s.t. $K = G^\theta$: fixed points

Example

$G = \mathrm{GL}_n(\mathbb{C})$

1. $\theta(g) = {}^t g^{-1} \rightsquigarrow K = \mathrm{O}_n(\mathbb{C})$: orthogonal grp
2. $\theta(g) = I_{p,q}\, g\, I_{p,q}$ $(I_{p,q} = \mathrm{diag}(1_p, -1_q))$
   $\rightsquigarrow K = \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$

Theorem

(G, K) : symmetric pair $\Longrightarrow \#K\backslash G/B < \infty$

$\therefore$ We can apply theory of conormal variety to $K \curvearrowright X := G/B$

We apply theory of conormal variety to $K \curvearrowright X := G/B$

- moment map:

$$\mu_X : T^*X = T^*(G/B) \xrightarrow{\mu_{G/B}} \mathfrak{g}^* \xrightarrow{\text{rest}} \mathfrak{k}^*$$

- Steinberg variety:

$$S_X = \mu_X^{-1}(0) = \mu_{G/B}^{-1}(\mathfrak{k}^\perp) = \mu_{G/B}^{-1}(\mathcal{N}(\mathfrak{s}))$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$: Cartan decomposition

$S_X = \bigcup_{\mathbb{O} \in K \backslash G/B} \overline{T^*_{\mathbb{O}}(G/B)} \xrightarrow{\mu_{G/B}} \mathcal{N}(\mathfrak{s})$: K-equivariant

$\mu_{G/B}(\overline{T^*_{\mathbb{O}}(G/B)}) = \overline{\mathcal{O}^K}$ induces $\Phi_X : K\backslash(G/B) \to \mathcal{N}(\mathfrak{s})/K$

orbit map:

$$\begin{array}{ccc} \Phi_X : K\backslash(G/B) & \longrightarrow & \mathcal{N}(\mathfrak{s})/K \\ \Cup & & \Cup \\ \mathbb{O} & \longrightarrow & \mathcal{O}^K \end{array}$$

$W(\mathcal{O}^K) := \Phi_X^{-1}(\mathcal{O}^K)$: Lusztig-Vogan cell carries Weyl group repr

Classical Steinberg variety is a special case of KGB theory

Example

$\mathbb{G} = G \times G, \quad \theta(g_1, g_2) = (g_2, g_1) \quad \rightsquigarrow \mathbb{K} = \Delta G = \{(g, g) \mid g \in G\}$

$\mathbb{B} = B \times B$: Borel subgroup for $\mathbb{G}$

$\mathbb{K} = \Delta G \curvearrowright X = \mathbb{G}/\mathbb{B} = G/B \times G/B$

$\rightsquigarrow \; G \curvearrowright X = G/B \times G/B$: diag action

$\mathfrak{s} = \{(z, -z) \mid z \in \mathfrak{g}\} \quad \rightsquigarrow \mathcal{N}(\mathfrak{s}) \simeq \mathcal{N}(\mathfrak{g})$

Parametrizing KGB for type A

Again we assume $G = \mathrm{GL}_n(\mathbb{C})$, i.e., type A case

Theorem

If $G = \mathrm{GL}_n(\mathbb{C})$, K-orbits $K\backslash(G/B)$ are parametrized by the set

$$\{(\mathcal{O}^K, C) \mid \mathcal{O}^K \in \mathcal{N}(\mathfrak{s}), C \in \mathrm{Irr}(\mathscr{B}_u)\ (u \in \mathcal{O}^K)\}$$

Let $\mathcal{O}^G_\lambda := G \cdot \mathcal{O}^K \in \mathcal{N}(\mathfrak{g})/G$ for some $\lambda \in \mathscr{P}(n)$
Then the intersection decomposes:

$$\mathcal{O}^G_\lambda \cap \mathfrak{s} = \bigcup_{i=1}^{m_K(\lambda)} \mathcal{O}^K_i \qquad (\mathcal{O}^K_i \in \mathcal{N}(\mathfrak{s})/K)$$

where $\{\mathcal{O}^K_i \mid 1 \le i \le m_K(\lambda)\}$ are nilpotent K-orbits, including original $\mathcal{O}^K$

Lemma

1. *$\mathcal{O}^K_i$ is a Lagrangian subvariety in $\mathcal{O}^G_\lambda$ of* $\dim = \frac{1}{2} \dim \mathcal{O}^G_\lambda$
2. *$\{\mathscr{B}_u \mid u \in \mathcal{O}^K_i\ (1 \le i \le m_K(\lambda))\}$ are all isomorphic*

As an applicatin, we get a formula for # of K-orbits on the flag variety $X = G/B$

Corollary

If $G = \mathrm{GL}_n(\mathbb{C})$,
$$\# K\backslash G/B = \sum_{\lambda \in \mathscr{P}(n)} m_K(\lambda) \cdot \dim \sigma_\lambda = \sum_{\lambda \in \mathscr{P}(n)} m_K(\lambda) \cdot \#\mathrm{STab}(\lambda)$$

Remark

For $G = \mathrm{GL}_n(\mathbb{C})$ and connected K,
$\# K\backslash G/B = \#$(irred HC $(\mathfrak{g}, K)$-modules with trivial central character)

A formula for $m_K(\lambda)$ and a combinatorial structure of the set $\{\mathcal{O}_i^K \mid 1 \leq i \leq m_K(\lambda)\}$ are obtained by the joint work with A. Wachi

To be continued ...