

相模原高校高大連携講座

一目見たら忘れられない証明

—Visual Proof の世界—

青山学院大学 理工学部物 理数理学科

西山 享

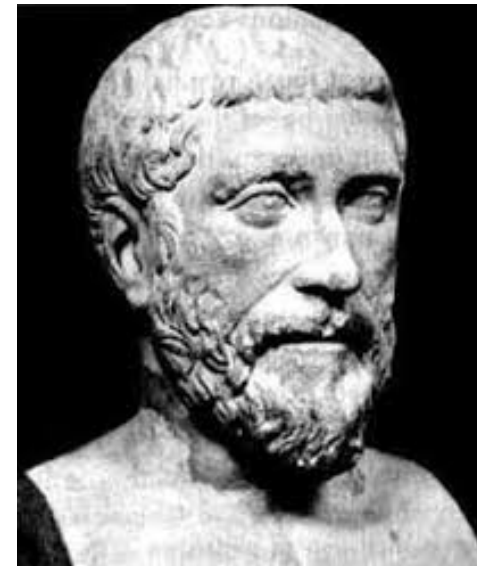
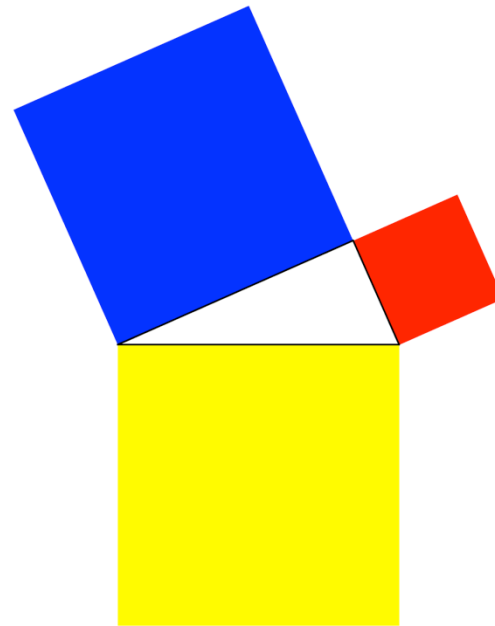
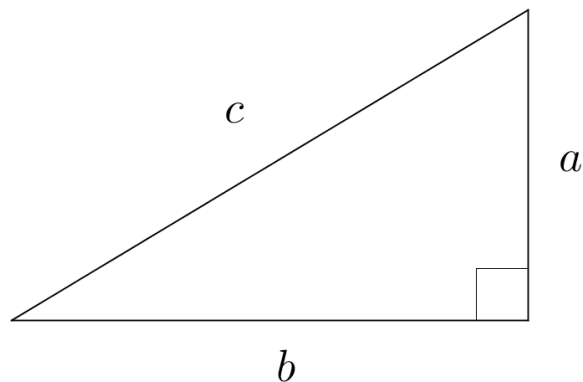
2014.11.15

Part 1

ウォーミングアップ

ピタゴラスの定理

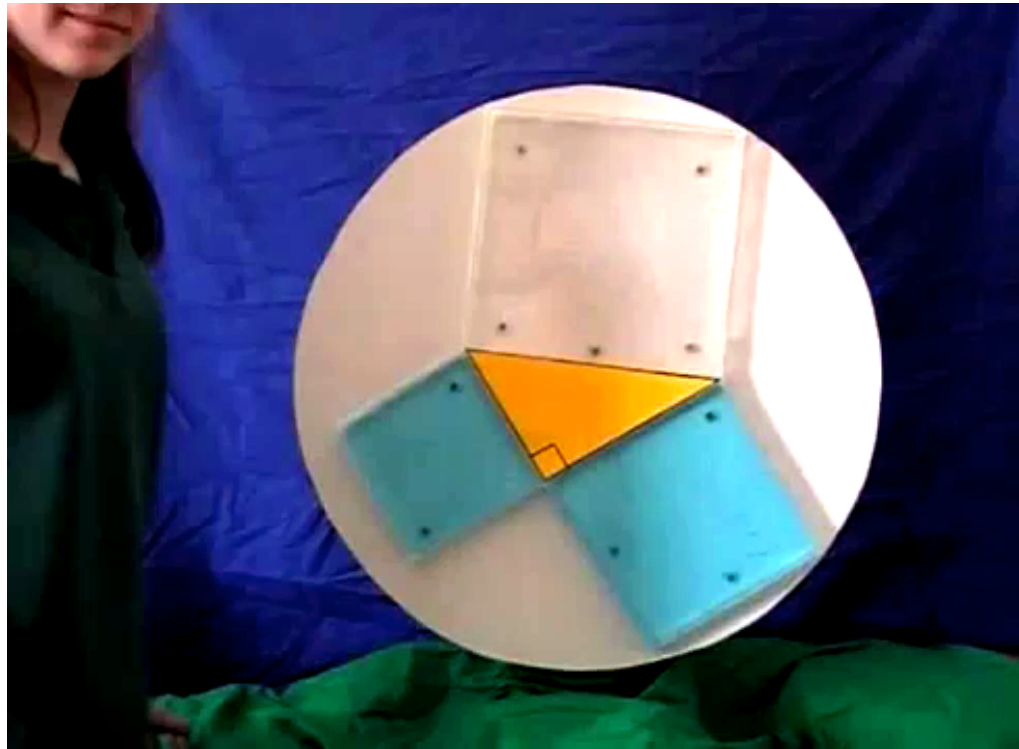
$$a^2 + b^2 = c^2$$



「万物は数である」

サモアのピタゴラス (前570??—前500??)

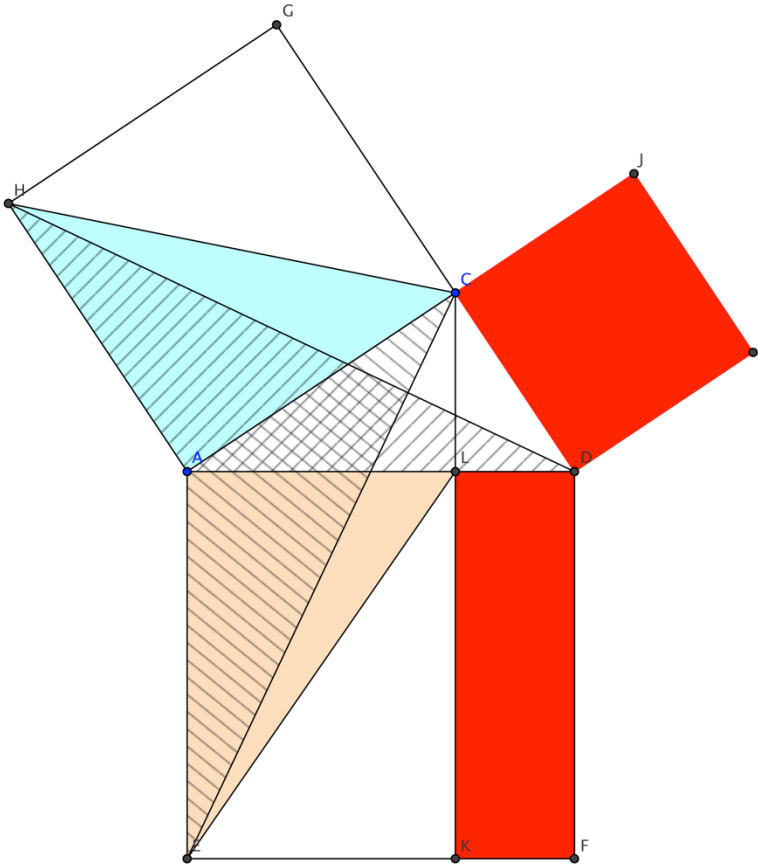
証明!!!



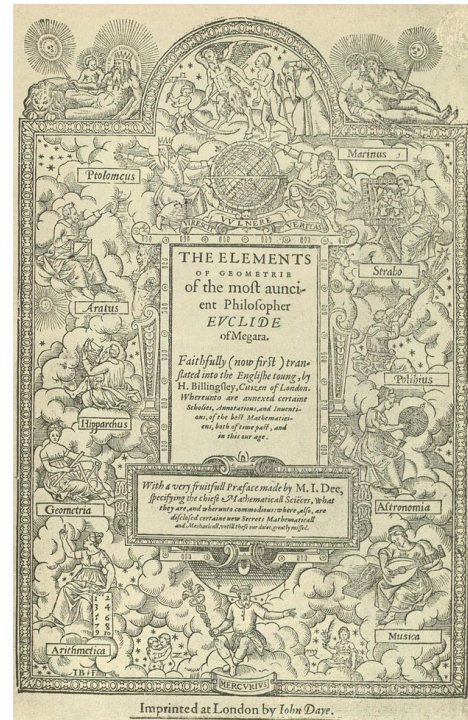
- Pythagorean theorem water demo
- <http://www.youtube.com/watch?v=CAkMUdeB06o>

ユークリッドの証明

アレキサンドリアのユークリッド
プトレマイオス1世の頃 (前306-283)



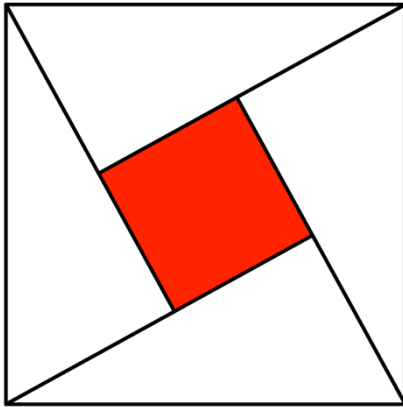
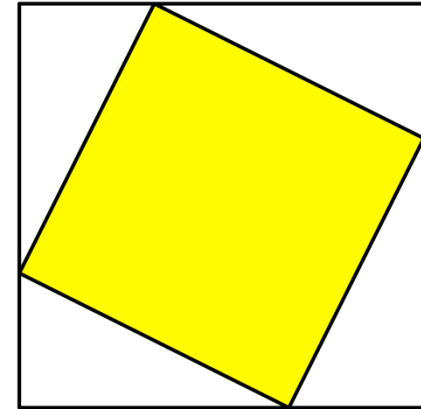
ユークリッド「原論」
(The Elements)



一目で分かる???証明 (ピタゴラスの定理)

(大正方形) = (小正方形(黄)) + 4×(直角三角形)

$$(a + b)^2 = c^2 + 4 \cdot \frac{1}{2} ab$$
$$a^2 + 2ab + b^2 = c^2 + 2ab$$

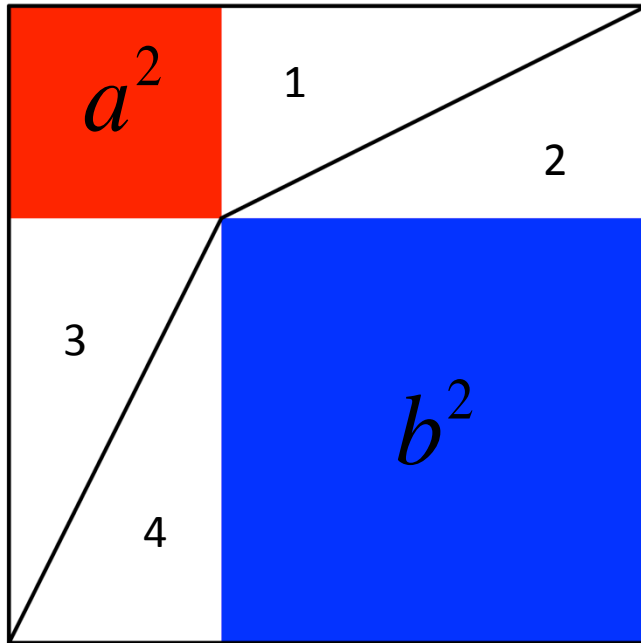


(大正方形) = (小正方形(赤)) + 4×(直角三角形)

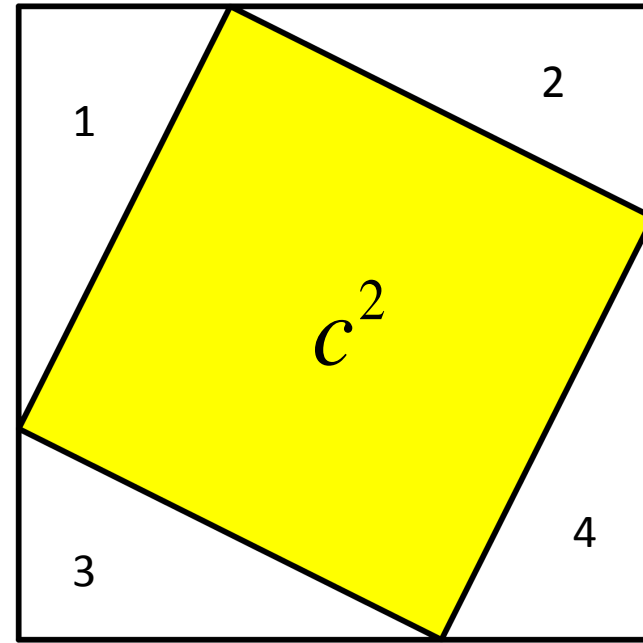
$$c^2 = (a - b)^2 + 4 \cdot \frac{1}{2} ab$$

$$c^2 = a^2 - 2ab + b^2 + 2ab = a^2 + b^2$$

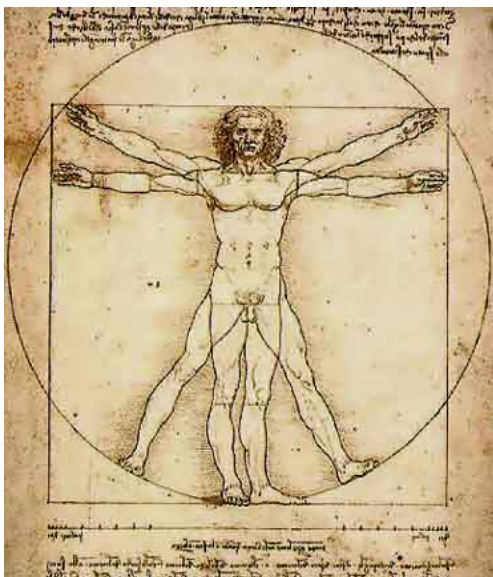
一目で分かる証明 (ピタゴラスの定理)



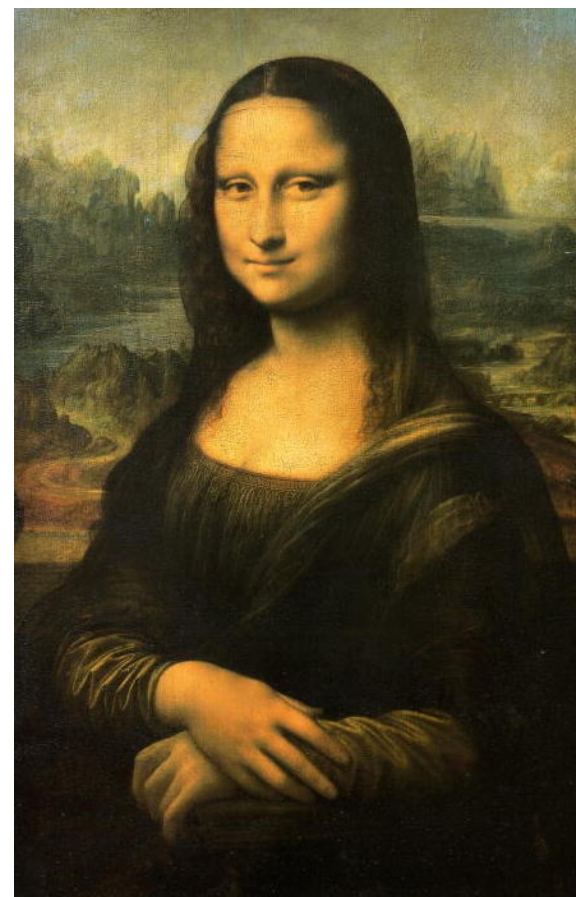
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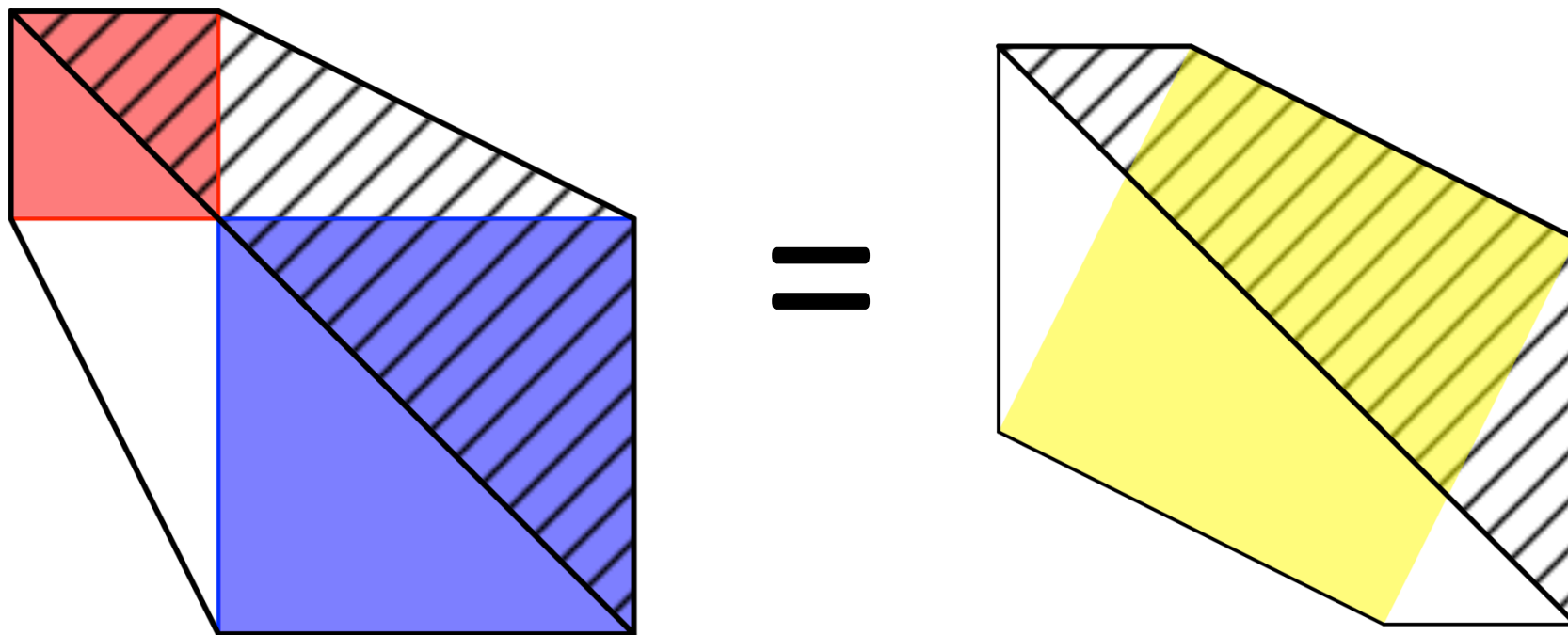
レオナルド・ダ・ヴィンチ の証明を味わう



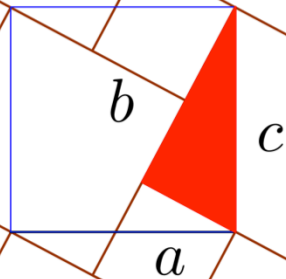
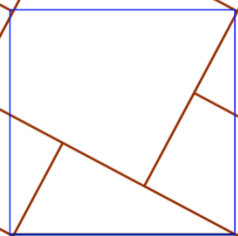
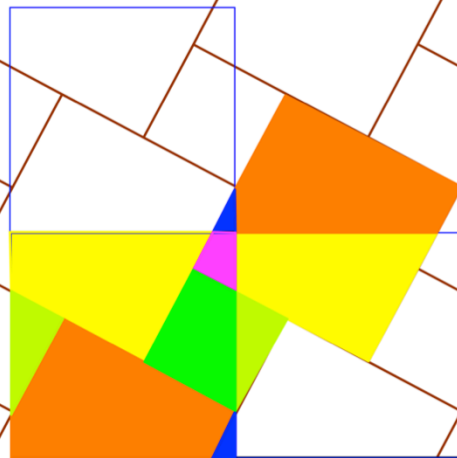
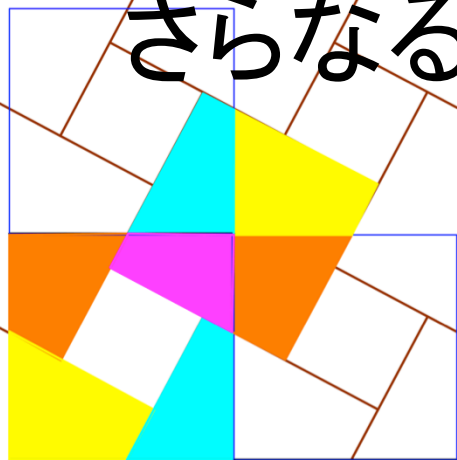
レオナルド・ダ・ヴィンチ
(1452 - 1519)



レオナルド・ダ・ヴィンチ の証明を味わう



さらなる証明



Part 2

べき乗和の公式

$$\sum_{k=1}^n k = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{ \frac{1}{2}n(n+1) \right\}^2$$

$$\sum_{k=1}^n k^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$\sum_{k=1}^n k^5 = 1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

$$\sum_{k=1}^n k^m = \cdots \text{(ベルヌーイ数を使って書ける)}$$

ベキ乗和 (1 乗和)

$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

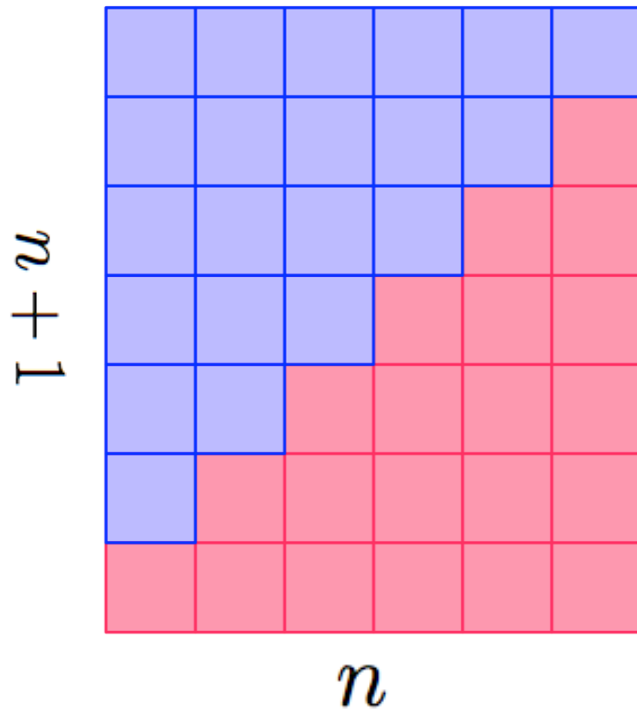
ガウス少年のひらめき



カール・フリードリッヒ・ガウス
(1777 – 1855)

ベキ乗和 (1 乗和)

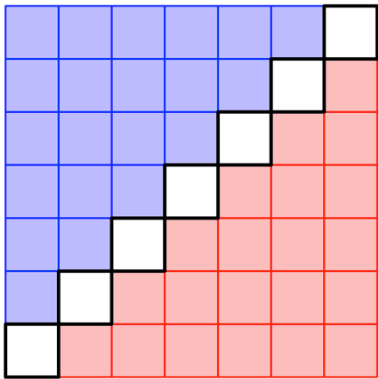
$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$



$$2 \sum_{k=1}^n k = (\text{長方形}) = n(n+1)$$

ベキ乗和 (1 乗和)

$$1 + 2 + 3 + \cdots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$



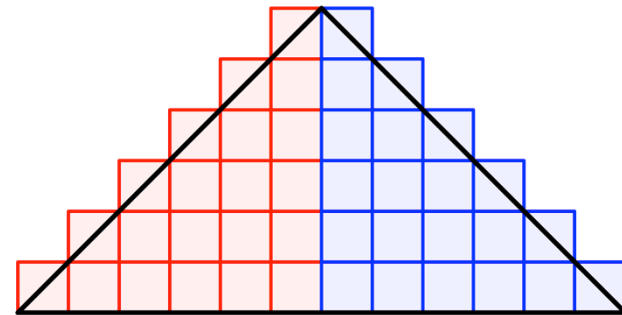
$$S := \sum_{k=1}^n k$$

$$(n+1)^2 = (\text{正方形}) = 2S + (\text{対角線}) = 2S + (n+1)$$

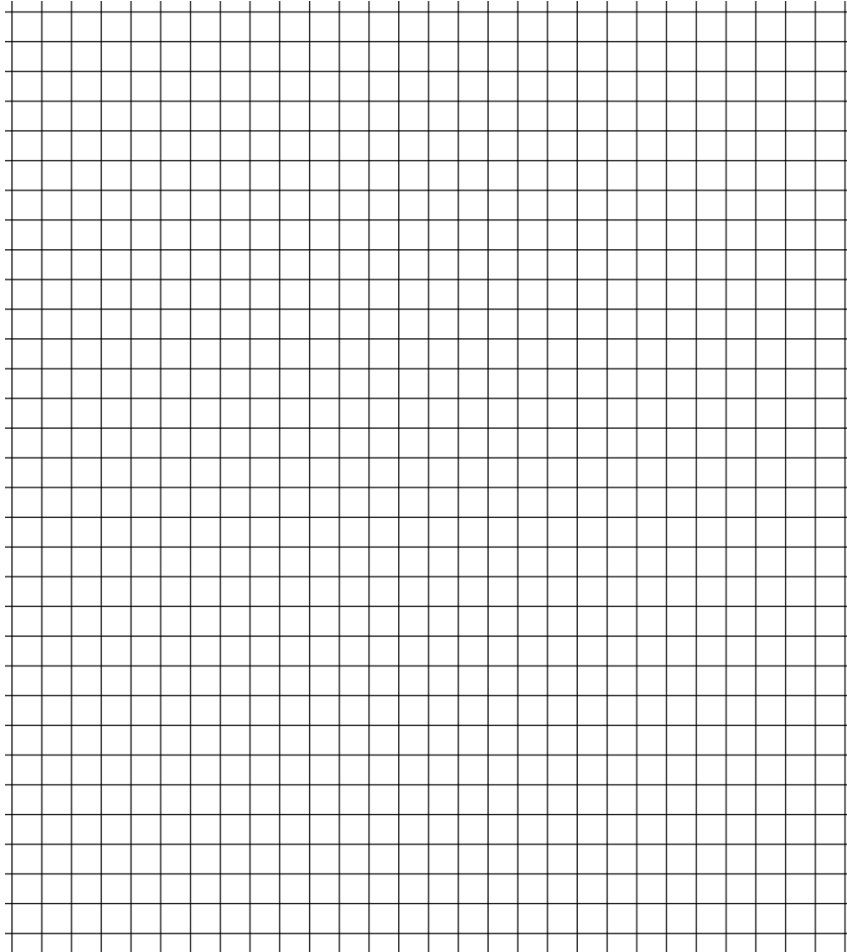
$$\therefore 2S = (n+1)^2 - (n+1) = (n+1)n$$

$$S := \sum_{k=1}^n k$$

$$\begin{aligned} 2S &= (\text{大三角形}) + n \times 2 \times (\text{小三角形}) \\ &= \frac{1}{2}(2n) \times n + n = n(n+1) \end{aligned}$$



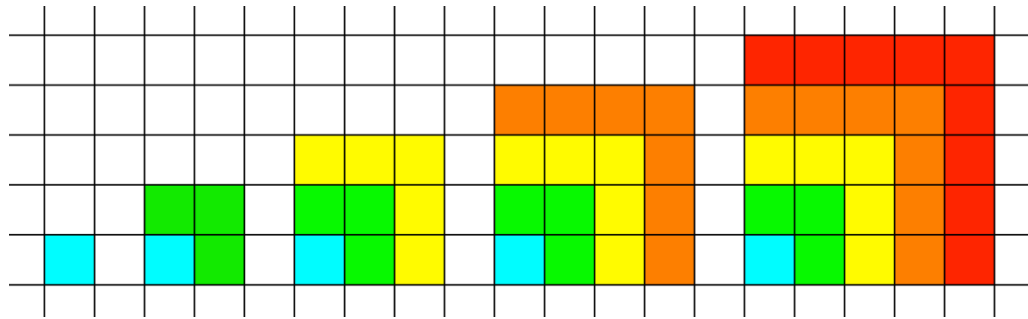
自分で工夫してみよう！



← 一目で分かる工夫？

たとえば・・・?!

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = \sum_{k=1}^n (2k - 1) = n^2$$

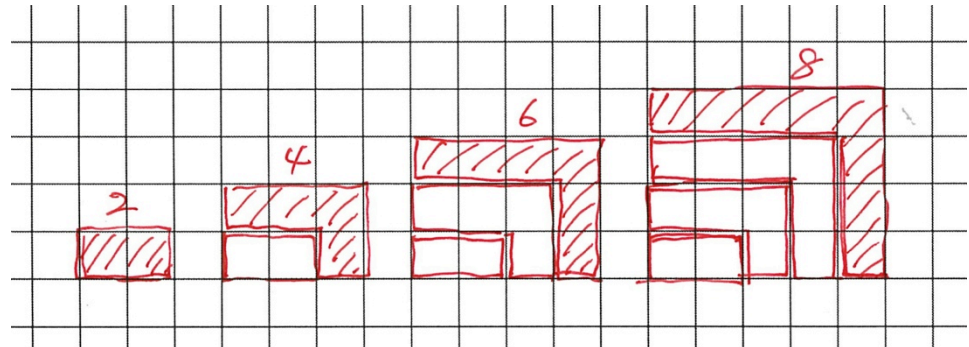


$$n^2 = \sum_{k=1}^n (2k - 1) = 2 \sum_{k=1}^n k - n$$

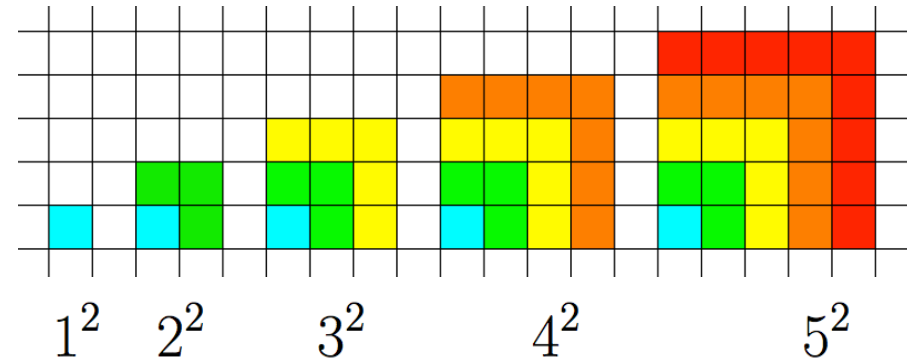
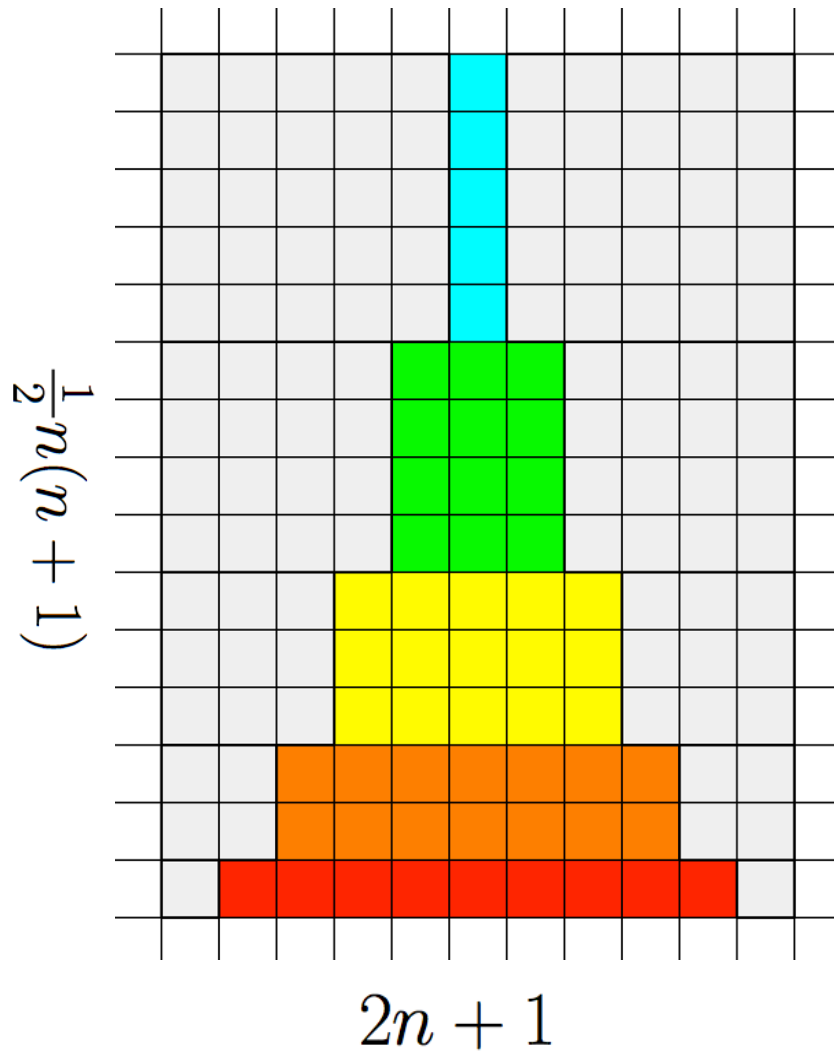
$$\therefore \sum_{k=1}^n k = \frac{n^2 + n}{2}$$

$$2 + 4 + 6 + \dots + 2n = \sum_{k=1}^n (2k) = n \times (n + 1)$$

$$\therefore 2 \sum_{k=1}^n k = n(n + 1)$$



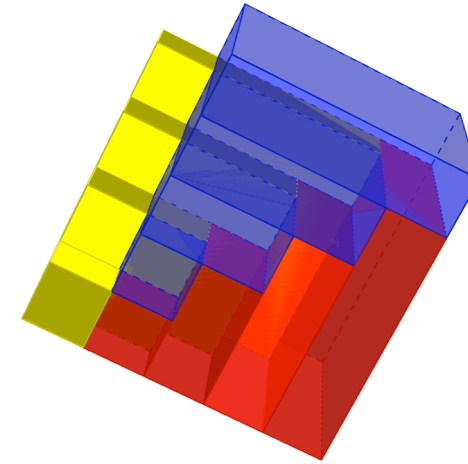
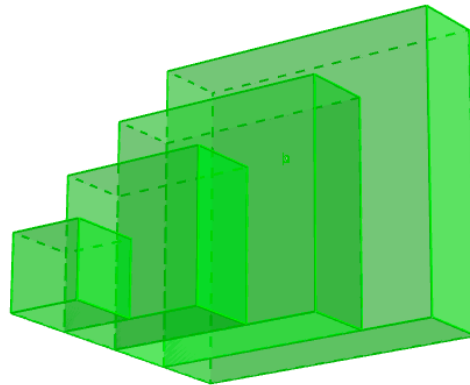
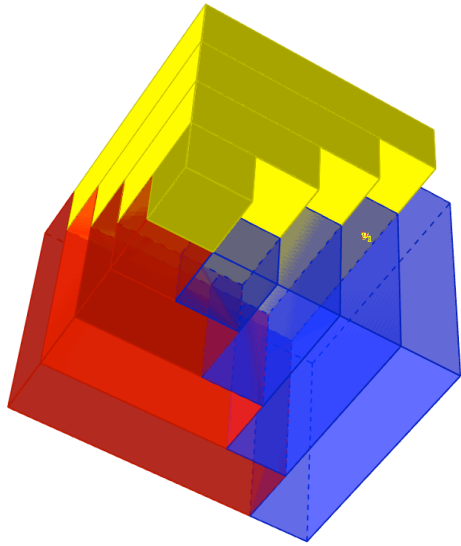
2乗和



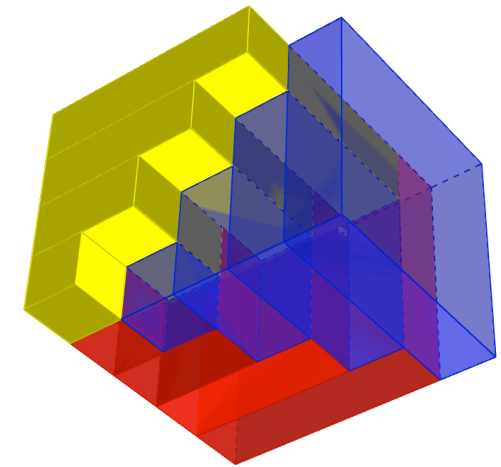
$$3 \times \sum_{k=1}^n k^2 = \frac{1}{2}n(n+1) \times (2n+1)$$

$$\therefore \sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

2乗和 その2

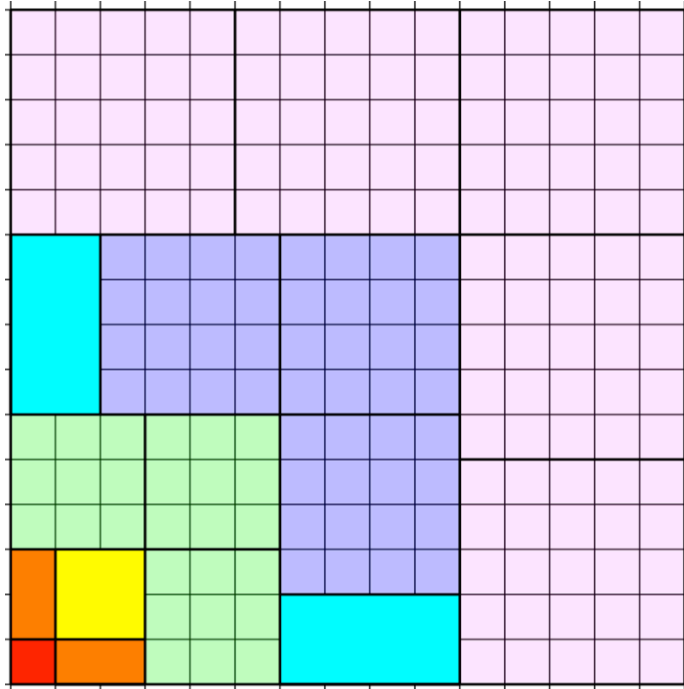


$$\begin{aligned} 3 \times \sum_{k=1}^n k^2 &= n \times n \times (n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)(2n+1) \end{aligned}$$



3乗和 !!

$$\frac{1}{2}n(n+1)$$



$$\frac{1}{2}n(n+1)$$

$$\sum_{k=1}^n k^3 = \left\{ \frac{1}{2}n(n+1) \right\}^2$$



その他の応用

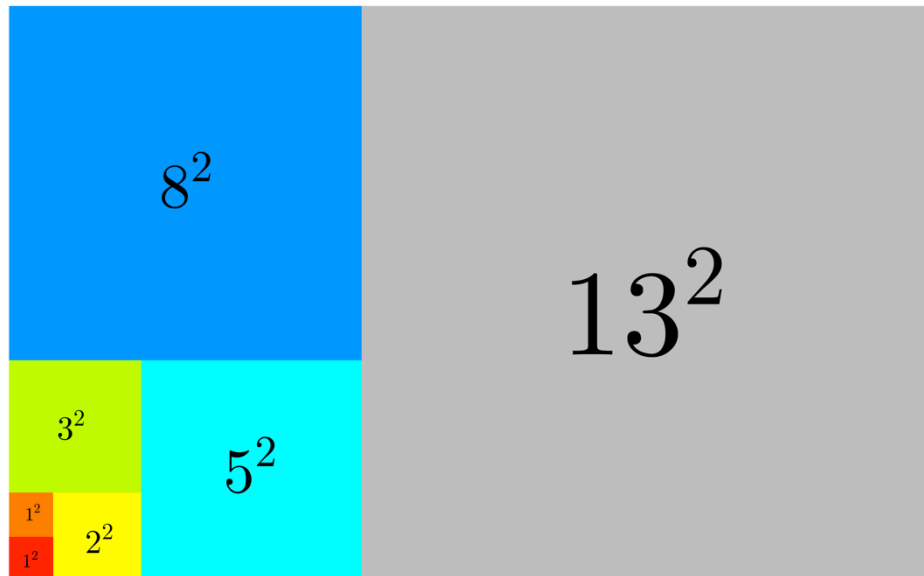


- フィボナッチ数列

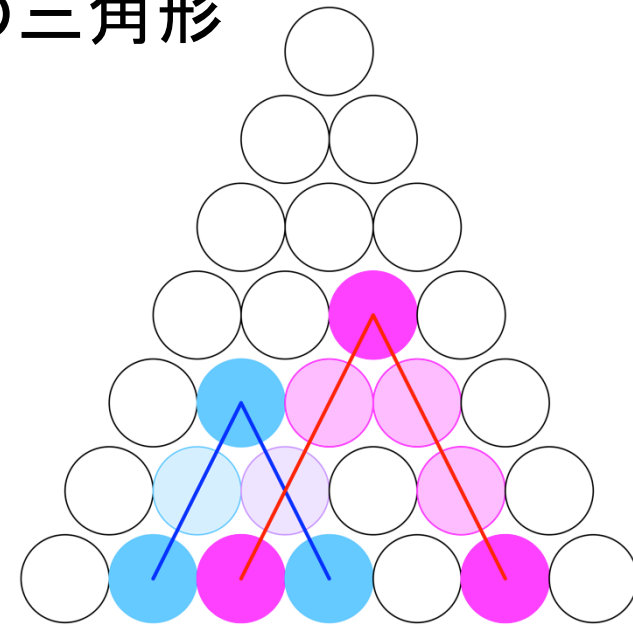
1, 1, 2, 3, 5, 8, 13, ...

$$F_{n+2} = F_{n+1} + F_n$$

$$F_1^2 + F_2^2 + F_3^2 + \dots + F_n^2 = F_n F_{n+1}$$



- パスカルの三角形



$${}_n C_2 = \frac{1}{2}n(n + 1)$$

第1部 おしまい