Infinitesimal twists along orbits

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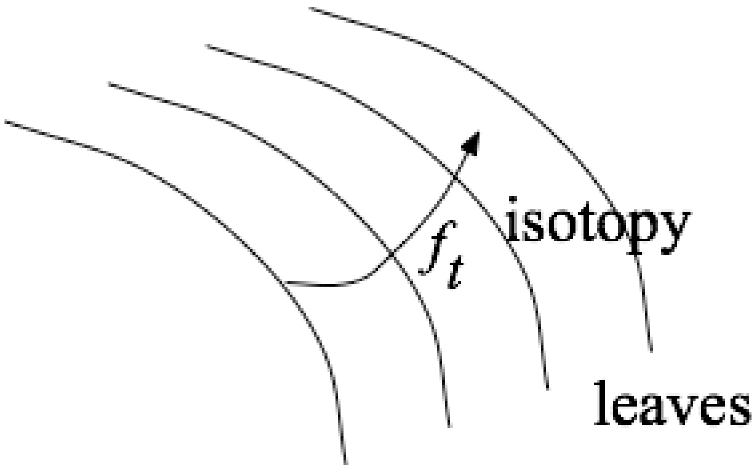
Contents

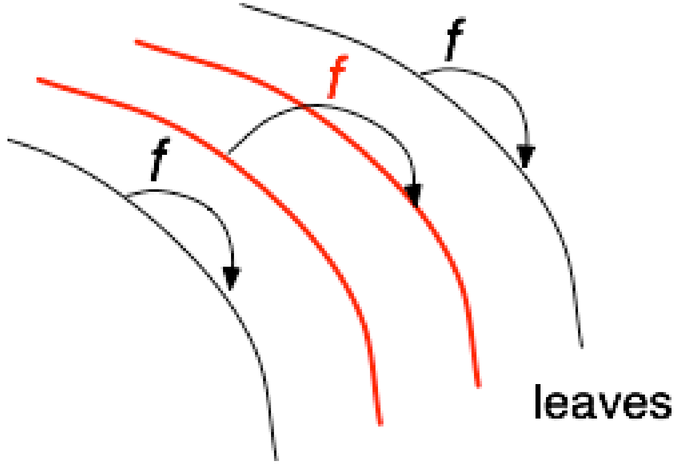
§1, Invariant foliations

In Le Calvez's talk



foliations dynamically transverse to the isotopy ⟺

In this talk



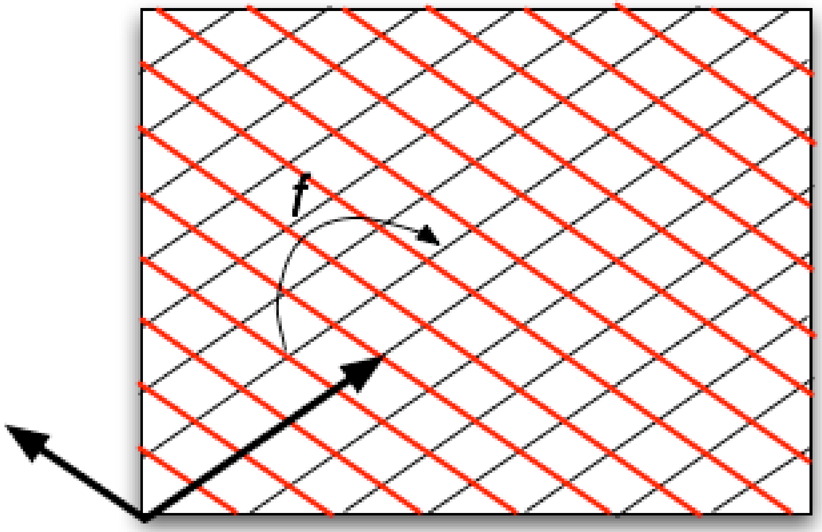
foliations invariant under diffeomorphisms

f maps each leaf onto a leaf

Examples of invariant foliations

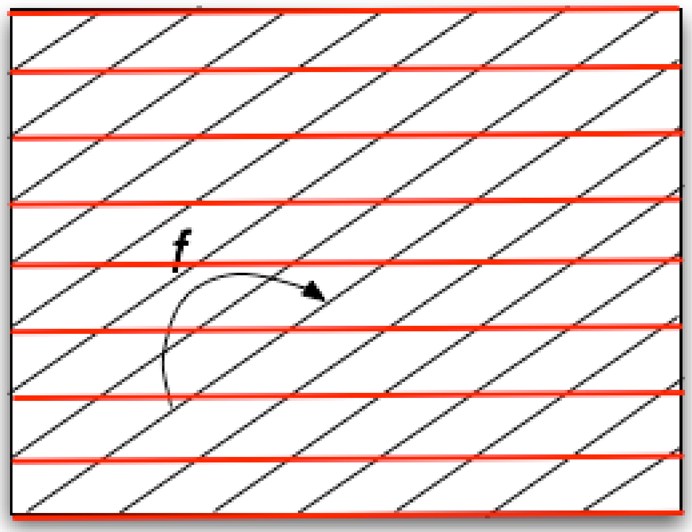
$$f : T^2 \to T^2 \text{ ; a diffeomorphism}$$

Anosov Diffeom.



$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Irrational transf.



$$f(x, y) = (x + \alpha, y + \beta)$$

$$\alpha / \beta \notin \mathrm{Q}, \alpha, \beta \notin \mathrm{Q}$$

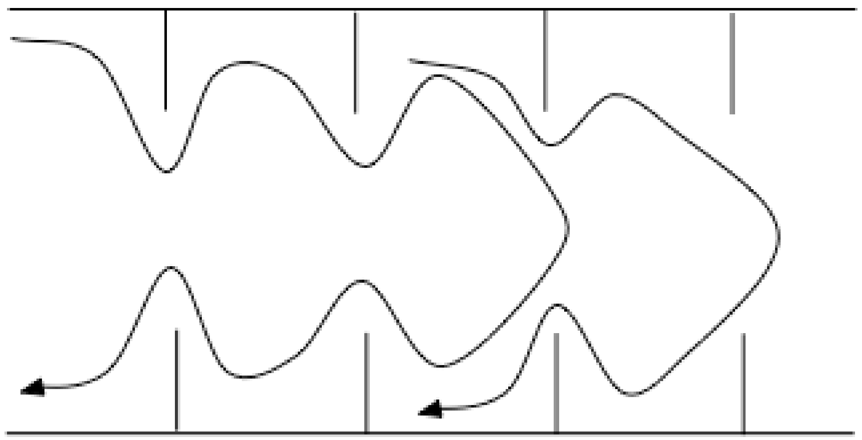
Assumption

In this talk, we will respect our attention to diffeomorphisms of the torus T^2

Remark (my original interest)

When can a homeomorphism of R^2 without a fixed point be embedded in a flow? (flowability)

> leaf preserving homeoms
> $\rightarrow$ foliation preserving diffeoms



A homeomorphism which can not be the time one map of a flow
(by M. Brown)

The other examples

(N–, F. Le Roux)

§ 2, Projectivized bundles

TT^2 : the tangent bundle of T^2

$PT^2 = \{(z,v) \in TT^2; v \neq 0\} / v \sim kv \quad (k \neq 0)$

projectivized bundle

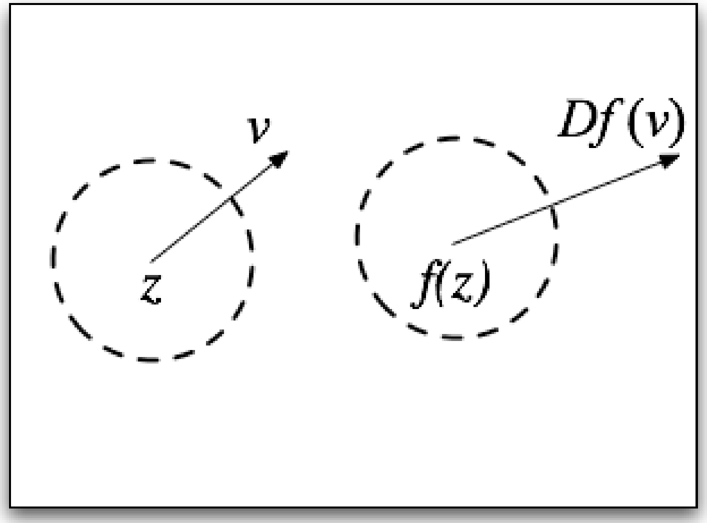
$f : T^2 \to T^2$: a diffeomorphism

$Df : TT^2 \to TT^2$: the derivative of f

$Pf : PT^2 \to PT^2$: the diffeom induced from Df

i.e. $(z, [v]) \in PT^2$,

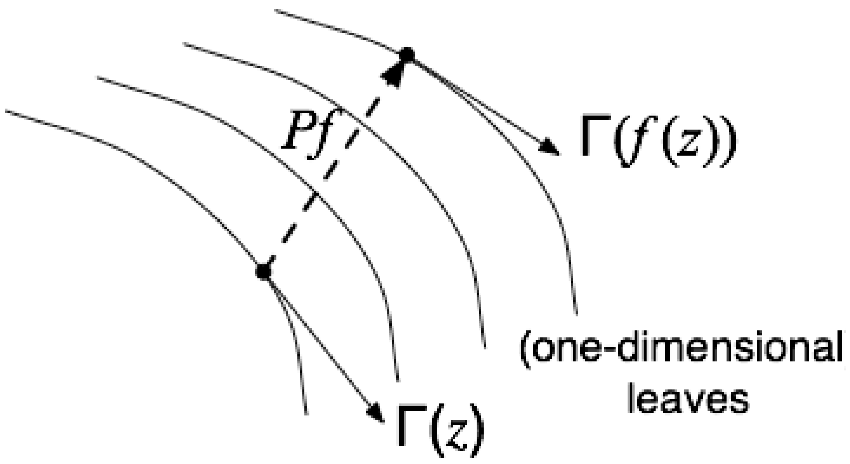
$Pf(z,[v]) = (z, [Df(v)])$

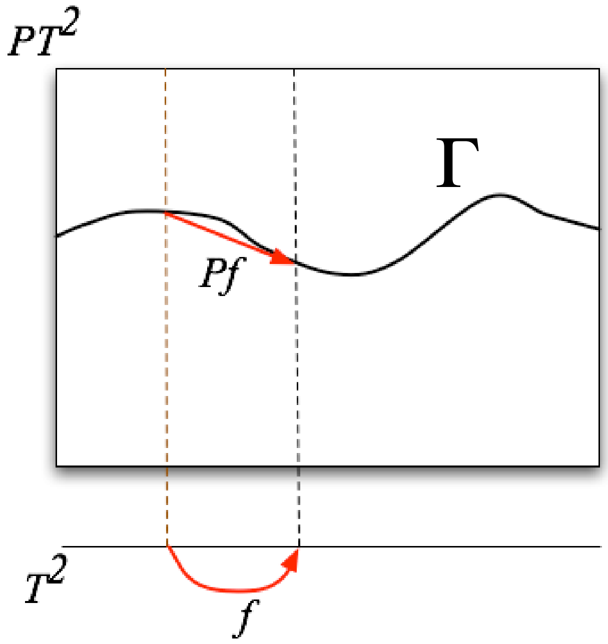


Lemma. f is tangent to a C^∞ foliation $\Im$

$\Leftrightarrow$ There is a C^∞ section $\Gamma : T^2 \to PT^2$

such that $Pf(\Gamma(z)) = \Gamma(f(z))$

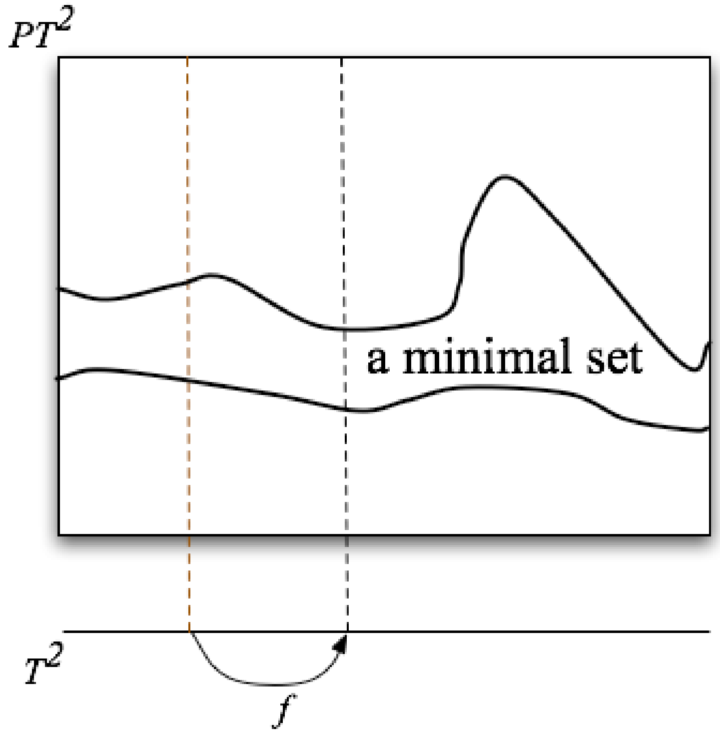




How to find such a section?

One of candidates is minimal sets of Pf

(i.e. closed Pf - invariant sets whih is minimal w.r.t. the inclusion)



§3, Model case

Def. f is tangentially distal

iff $\inf\left\{\left\|Df^n(v)\right\|; n \in \mathbb{Z}\right\} \neq 0$ for any $v \neq 0$

Theorem. (Shigenori Matsumoto, N-, 1997)

If $f : T^2 \to T^2$ is tangentially distal and minimal

(i.e. all orbits dense),

then there is a C^0 1-dim foliation tangent to f.

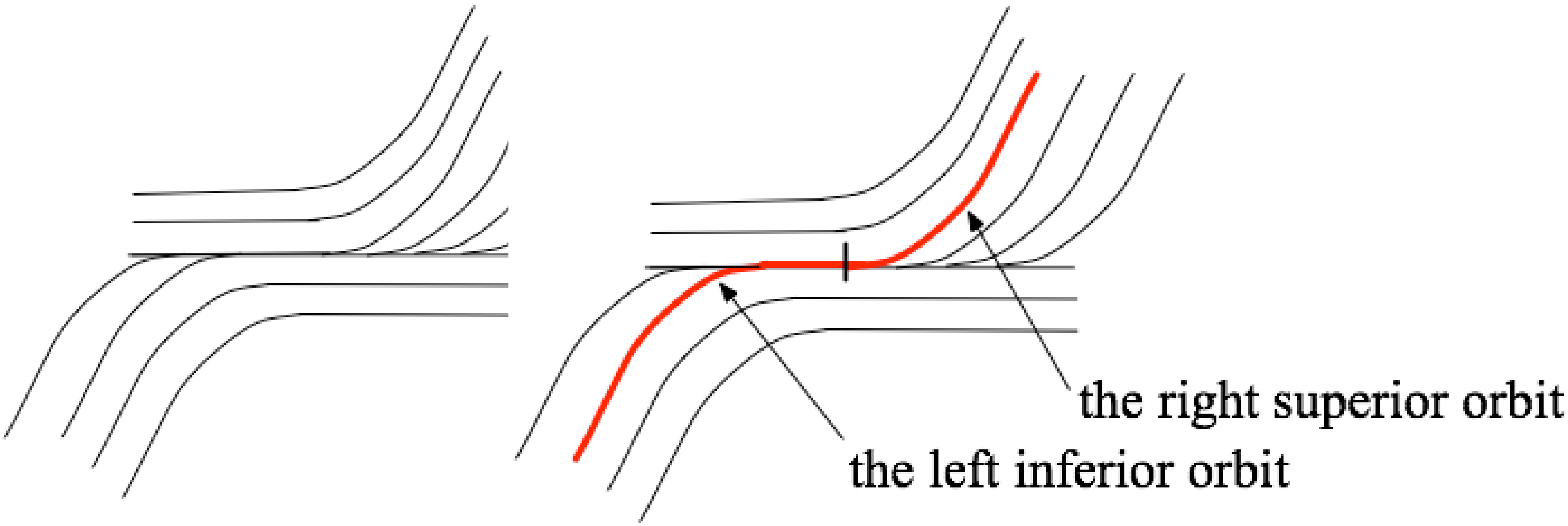
Sketch of proof

1) To find an invariant C^0 section for Pf.

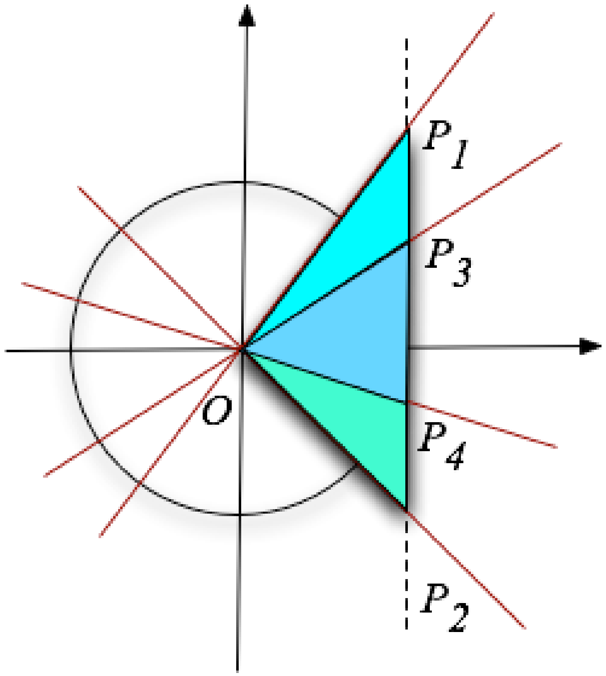
$\rightarrow$ routine work

2) To find a tangent C^0 foliation.

(not always uniquely integrable)



§4, Properties of PSL(2,R)



Cross ratio for straight lines

$$(P_1,P_2,P_3,P_4) = \frac{\overline{P_1P_3}}{\overline{P_3P_2}} \Big/ \frac{\overline{P_1P_4}}{\overline{P_4P_2}}$$

The cross ratio is invariant under $SL(2,R)$ because $SL(2,R)$ preserves the area of the triangles

$$\frac{\overline{OP_1P_3}}{\overline{OP_3P_2}} \Big/ \frac{\overline{OP_1P_4}}{\overline{OP_4P_2}}$$

How to use the cross ratio.

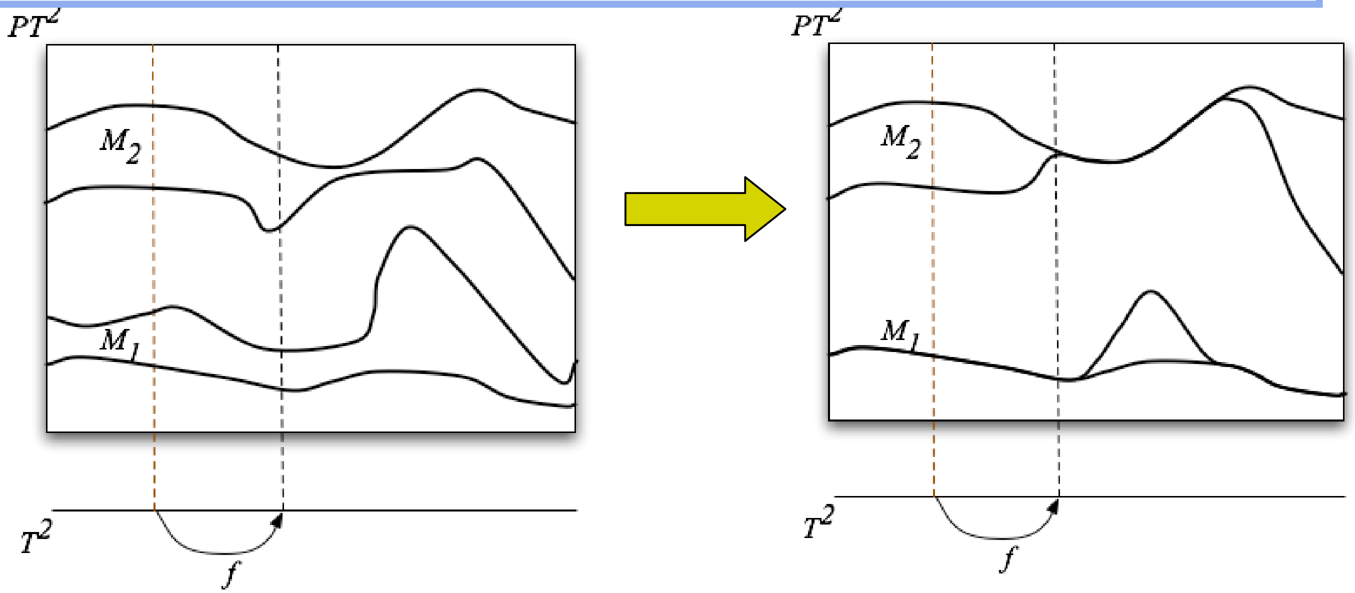
Here we consider the case when

Pf has two minimal sets M_1, M_2.

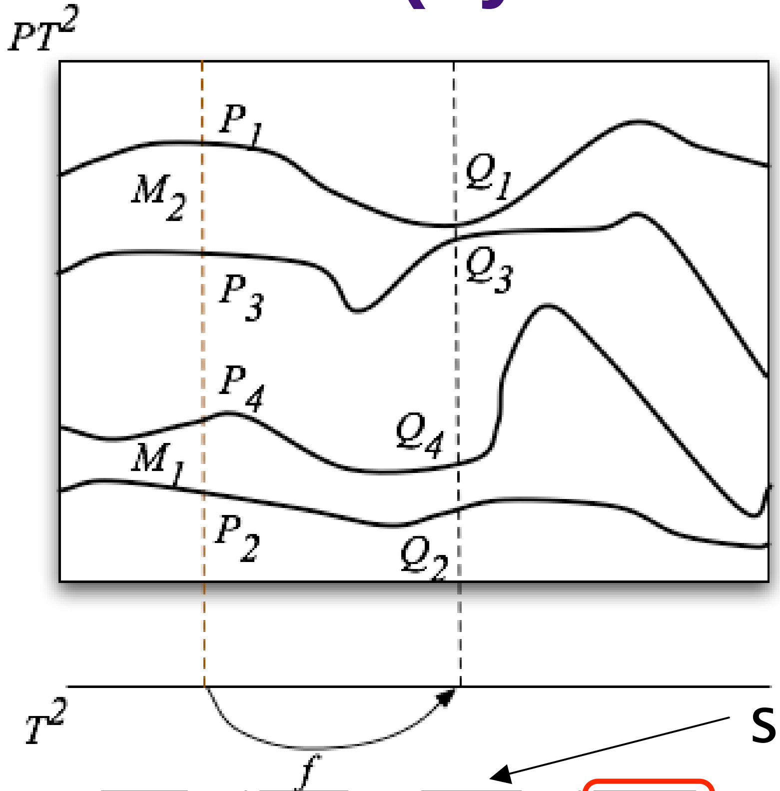
i.e. there are closed invariant sets M_i $(i = 1,2)$ in PT^2

which are minimal among such closed invariant sets.

Lemma. For any fiber $\{z\} \times P^1$, either $M_1 \cap (\{z\} \times P^1)$ or $M_2 \cap (\{z\} \times P^1)$ consists of a single point.



Proof of Lemma (by absurdity)



small

bounded below

cross ratio $\quad \dfrac{\overline{P_1P_3}}{\overline{P_3P_2}} \Big/ \dfrac{\overline{P_1P_4}}{\overline{P_4P_2}} = \dfrac{\overline{Q_1Q_3}}{\overline{Q_3Q_2}} \Big/ \dfrac{\overline{Q_1Q_4}}{\overline{Q_4Q_2}} \to 0 \quad$ contradiction

By the minimality of M_2, $\overline{Q_1Q_3}$ approaches to 0.

On the other hand, $\overline{Q_2Q_3}$ and $\overline{Q_1Q_4}$ are bounded below. ∎

Then we can

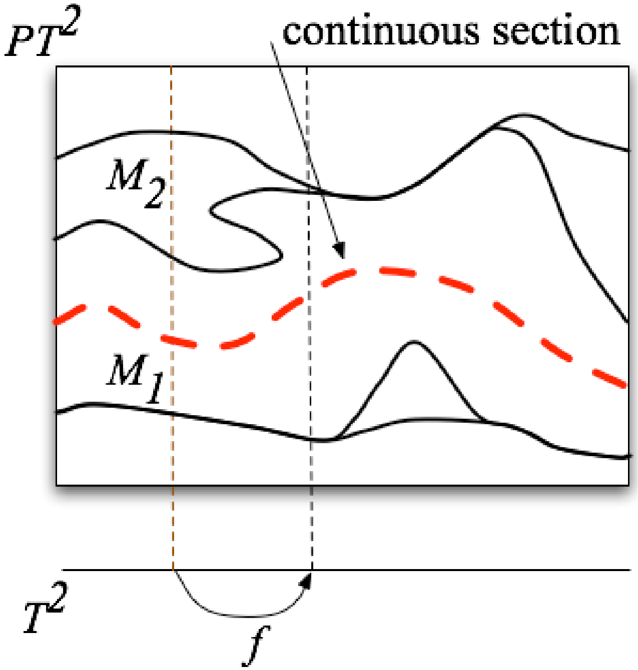
1) find a continuous section between M_1 and M_2.

(N - and Noda, 2005)

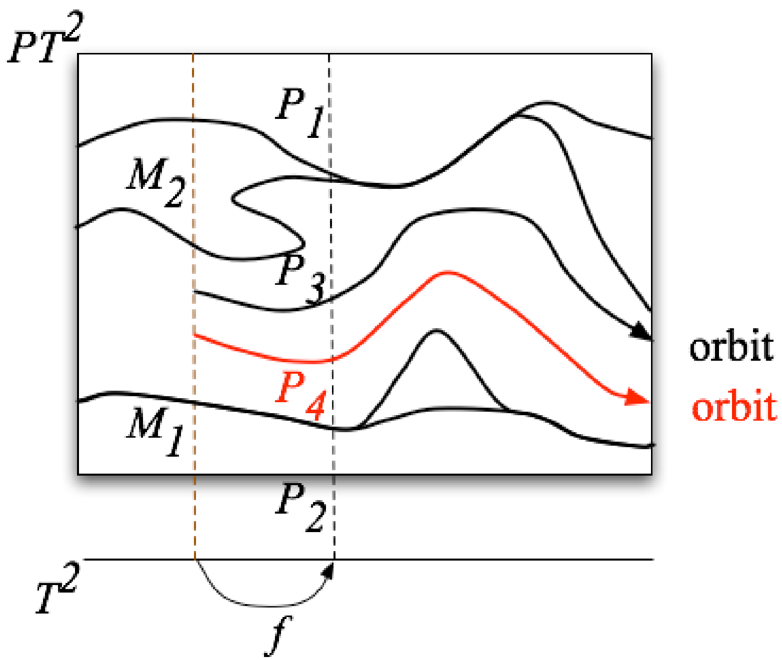
2) control all the orbits between M_1 and M_2

by a single orbit between M_1 and M_2. (N-,2007)

1)



2)



§5, Infinitesimal twist along orbits

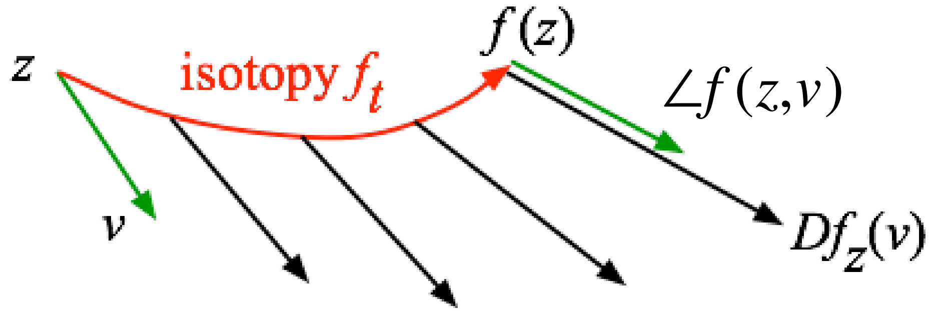
$f : T^2 \to T^2$; a C^2 diffeomorphism isotopic to id

f_t is its isotopy ($f_0 = \mathrm{id}, f_1 = f$)

$T_1 T^2$: the unit tangent bundle of T^2

$\angle f : T_1 T^2 \to T_1 T^2$; a diffeomorphism

$$\text{defined by } \angle f(z,v) = \left(f(z), \frac{Df_z(v)}{\left\| Df_z(v) \right\|} \right)$$



§6, Ruelle invariant

$\tilde{f} : T^2 \times \mathrm{R} \to T^2 \times \mathrm{R}$: the lift of $\angle f$

with respect to the isotopy f_t

$$\rho(z) = \lim_{n\to\infty} \frac{\tilde{f}^n(z,0)}{n} \text{ if it exists}$$

Def. μ : an f - invariant prob. measure of T^2

The Ruelle invariant

$$R_\mu(f) := \int_{T^2} \rho(z) d\mu$$

" Average of the twist along the orbits"

Another def of Ruelle invariant

by Ruelle

$G = SL(2,\mathrm{R})$

$\tilde{G}$: the universal cover of G

Let A be an element of $\tilde{G}$.

i.e. $A(t) \in G, A(0) = e \quad (0 \le \mathrm{t} \le 1)$

We will define the angle of A

Polar decomposition of $A(t)$

$$A(t) = O(t)S(t)$$

where $S(t)$: (positive) symmetric matrix

$O(t)$: orthogonal matrix

$$(S(t) = \sqrt{{}^{T}A(t)A(t)}, O(t) = A(t)S_t^{-1})$$

$$\theta(t) := (\text{the angle of } O(t)) \in S^1$$

$$O(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

$$\Theta(A) \in \mathrm{R}; \text{ the variation of } \theta(t)$$

For $n \in \mathrm{Z}_+$

$$A(z,n) := \left(t \mapsto \frac{Df_{nt}(z)}{\sqrt{\det Df_{nt}(z)}} \right) \in \tilde{G} \quad (0 \le t \le 1)$$

Lemma. $\rho(z) = \lim_{n\to\infty} \dfrac{\Theta(A(z,n))}{n}$

The Ruelle invariant $R_\mu(f) = \int_{T^2} \lim_{n\to\infty} \frac{\Theta(A(z,n))}{n} d\mu$

Remark. The Ruelle invariant can be defined for symplectic matrices (by Ruelle).

Another def of Ruelle invariant

by Inaba, N-

μ : an f - invariant measure of T^2

$\pi : T_1 T^2 \to T^2$; the projection

Then there is a measure ν on $T_1 T^2$

s.t. $(\angle f)_* \nu = \nu$ and $\pi_* \nu = \mu$

$\Delta : T_1 T^2 \to \mathrm{R}$ defined by

$$\tilde{f}(z,v) = (f(z), v + \Delta f(z,v))$$

for $(z,v) \in T^2 \times \mathrm{R}$

$$\begin{array}{ccc}
T^2 \times \mathrm{R} & \xrightarrow{\tilde{f}} & T^2 \times \mathrm{R} \\
\downarrow & & \downarrow \\
T_1 T^2 & \xrightarrow{\angle f} & T_1 T^2 \\
\downarrow & & \downarrow \\
T T^2 & \xrightarrow{Df} & T T^2 \\
\downarrow & & \downarrow \\
T^2 & \xrightarrow{f} & T^2
\end{array}$$

Theorem (T. Inaba and N-, 2004)

$$R_\mu(f) = \int_{T_1 T^2} \Delta f \, d\nu$$

Outline of proof

By the disintegration theorem,

there is a prob. measure ν_z on each fiber $\{z\} \times S^1$

s.t. $\int_{T_1 T^2} \varphi(z,v)\, d\nu = \int_{T^2} d\mu \int_{\{z\}\times S^1} \varphi(z,v)\, d\nu_z$

for a continuous function φ.

$$\therefore \lim_{n\to\infty} \frac{1}{n} \int_{T_1 T^2} \Delta f^n(z,v)\, d\nu = \lim_{n\to\infty} \frac{1}{n} \int_{T^2} d\mu \int_{\{z\}\times S^1} \Delta f^n(z,v)\, d\nu_z$$

$$\therefore \lim_{n\to\infty} \frac{1}{n} \int_{T_1 T^2} \sum_{i=0}^{n-1} \Delta f(\tilde{f}^i(z,v))\, d\nu = \lim_{n\to\infty} \frac{1}{n} \int_{T^2} \Delta f^n(z,0)\, d\mu$$

$$\int_{T_1 T^2} \Delta f(z,s) d\nu = \int_{T^2} \rho(z) d\mu$$

$\blacksquare$

Theorem (Shigenori Matsumoto and N-, 2002)

$f : T^2 \to T^2$; a C^∞ - diffeomorphism isotopic to id

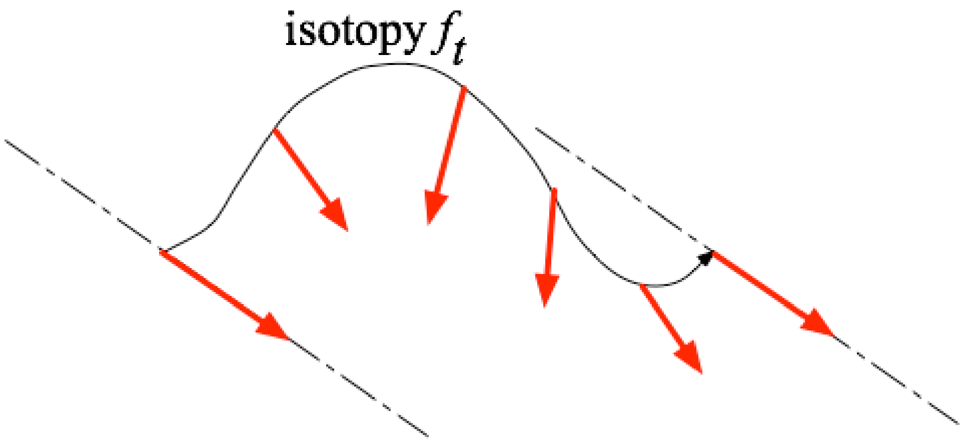
$\Rightarrow$ there is an f - invariant prob. measure μ

such that $R_\mu(f) = 0$

Key lemma.

$f : T^2 \to T^2$; a C^∞ - diffeomorphism isotopic to id.

$\Rightarrow$ $\Delta f(z,v) = 0$ for some point $(z,v) \in T_1 T^2$



Proof of (Key lemma→Theorem)

There is (z_n, v_n) s.t. $\Delta f^{n}(z_n, v_n) = 0$

Thus $\sum_{i=0}^{n-1} \Delta f(\tilde{f}^{i}(z_n, v_n)) = 0$

$$\nu_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta(\tilde{f}^{i}(z_n, v_n))$$

where δ is the Dirac measure at $\tilde{f}^{i}(z_n, v_n)$

ν : accumulation of ν_n

Then $R_\mu(f) = \int \Delta f(z,s)\, d\nu$

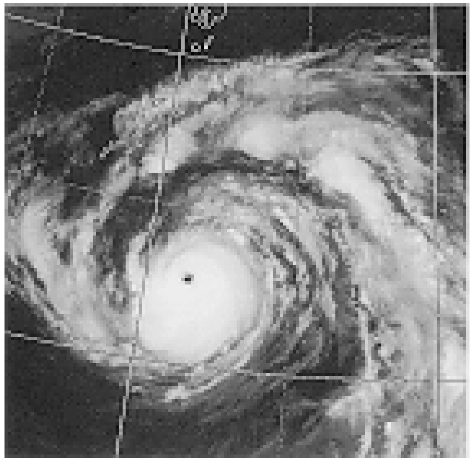
$$= \lim_{n\to\infty} \int \Delta f(z,s)\, d\nu_n$$

$$= \lim_{n\to\infty} \int \Delta f(z,s)\, \tfrac{1}{n} \sum_{i=0}^{n-1} \delta(\tilde{f}^{i}(z_n, v_n))$$

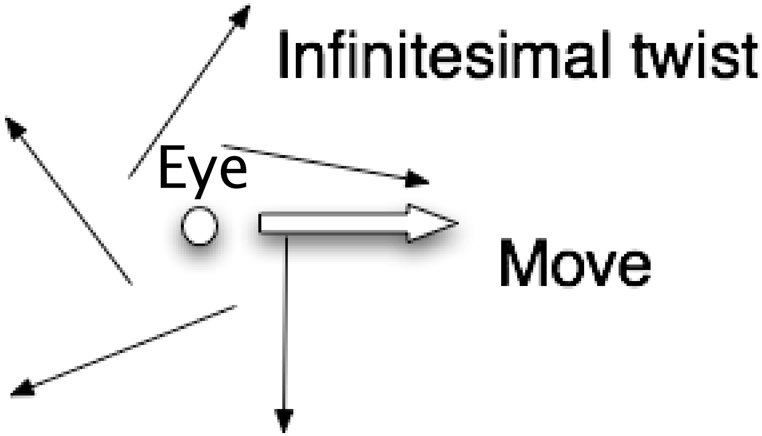
$$= \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} \Delta f(\tilde{f}^{i}(z_n, v_n)) = 0$$

∎

§7, “Eyes of typhoons”



from「気象庁ホームページ」



When does an Eye of typhoon turn out?

strong twist + slow move

How to describe this situation?

For $t \in \mathrm{R}$, $f_t = f_{t-[t]} \circ f^{[t]}$

$\overline{f_t} : \mathrm{R}^2 \to \mathrm{R}^2$; the lift of f_t with respect to f_t

For $x \in \mathrm{R}^2$,

$$K_n(x) = \max\left\{ \frac{\left\| (D\overline{f_n})_y - (D\overline{f_n})_x \right\|}{\left\| y - x \right\|} ; x \neq y, y \in R^2 \right\} \text{ for } x \in R^2$$

$S_n(x)$: symmetric part of the polar decomposition for $D\overline{f_n}(x)$

Theorem.

$f: T^2 \to T^2$; a C^∞ - diffeomorphism isotopic to id

s.t. f has no periodic point and

the Ruelle invariant $R_\mu(f) > 0$

$$N := \left[\frac{3\pi}{R_\mu(f)} \right] + 1$$

If $\| S_n(x) - \mathrm{id} \| < 1/2$ for any x and $1 \le n \le N$,

then there is a point x_0 s.t.

$$\frac{\sqrt{2}}{32 K_n(x_0)} \le d(x_0, f_n(x_0))$$