Theta lifting of two-step nilpotent orbits for the pair $O(p,q) \times Sp(2n,\mathbb{R})$ *

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Introduction

Let G be a linear reductive Lie group which is a subgroup in its complexification $G_{\mathbb{C}}$. We denote the Lie algebra of G by $\mathfrak{g}_0$, and its complexification by $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$. We will use the similar notation for any Lie group L; thus, $L_{\mathbb{C}}$ denotes its complexification, $\mathfrak{l}_0$ its Lie algebra, and $\mathfrak{l}$ the complexification of $\mathfrak{l}_0$.

Take a maximal compact subgroup K of G. Then K determines a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ and its complexification $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The adjoint action of $K_{\mathbb{C}}$ preserves $\mathfrak{s}$, and the set of all nilpotent elements $\mathcal{N}_{\mathfrak{s}}$ in $\mathfrak{s}$. It is well known that $\mathcal{N}_{\mathfrak{s}}$ is a normal variety and that it has finitely many $K_{\mathbb{C}}$-orbits ([2]).

Now consider a dual pair $(G, G') = (O(p,q), Sp(2n,\mathbb{R}))$ (see [1] for the properties of dual pairs). In this note, we define certain double fibration maps of nilpotent varieties for $O(p,q)$ and $Sp(2n,\mathbb{R})$. We use the double fibration maps to get a correspondence between nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$ and nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{s}'$, which is called a "theta lift". We describe the theta lifts of two-step nilpotent orbits in $\mathcal{N}_{\mathfrak{s}'}$, where $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ is a Cartan decomposition for $G' = Sp(2n,\mathbb{R})$ (Proposition 1.3).

If a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} \subset \mathfrak{s}$ is the theta lift of a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathfrak{s}'$, it is interesting to describe the regular function ring $\mathbb{C}[\overline{\mathcal{O}}]$ by means of $\mathbb{C}[\overline{\mathcal{O}'}]$. Our main results are descriptions of the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathcal{O}}]$ in terms of the double fibration maps (Theorem 2.4 and Proposition 3.4). In the course of the proof, we realize the closure $\overline{\mathcal{O}}$ of the orbit as a geometric quotient of the fiber of $\overline{\mathcal{O}'}$ (Proposition 3.3). As an application of these results, we get a formula of branching coefficients between different kind of classical groups (Corollary 3.5).

The $K_{\mathbb{C}}$-module structures of nilpotent orbits may reflect the K-type decompositions of the corresponding admissible representation of G via orbit method (or geometric quan-

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tization). Thus we can expect to extract information on the admissible representations from the geometry of nilpotent orbits. This will be treated elsewhere.

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1 Double fibration of nilpotent varieties

Let $G = O(p,q)$ be an orthogonal group of signature (p,q). Then a maximal compact subgroup K is isomorphic to $O(p) \times O(q)$. We realize them as follows.

$$G = O(p,q) = \left\{ g \in GL(p+q,\mathbb{R}) \mid {}^t g 1_{p,q} g = 1_{p,q} \right\}, \quad 1_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix},$$
$$K = O(p) \times O(q) = \begin{pmatrix} O(p) & 0 \\ 0 & O(q) \end{pmatrix}.$$

Then the corresponding (complexified) Cartan decomposition is given by

$$\begin{aligned} \mathfrak{g} &= \left\{ \begin{pmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{pmatrix} \mid \begin{array}{l} \alpha \in \mathrm{Alt}_p(\mathbb{C}) \\ \gamma \in \mathrm{Alt}_q(\mathbb{C}) \end{array}, \beta \in M_{p,q}(\mathbb{C}) \right\} \\ &= \begin{pmatrix} \mathrm{Alt}_p(\mathbb{C}) & 0 \\ 0 & \mathrm{Alt}_q(\mathbb{C}) \end{pmatrix} \oplus \begin{pmatrix} 0 & M_{p,q}(\mathbb{C}) \\ {}^t M_{p,q}(\mathbb{C}) & 0 \end{pmatrix} = \mathfrak{k} \oplus \mathfrak{s}. \end{aligned}$$

Hence we identify $\mathfrak{s}$ with $M_{p,q}(\mathbb{C})$ via

$$M_{p,q}(\mathbb{C}) \ni \beta \leftrightarrow \begin{pmatrix} 0 & \beta \\ {}^t\beta & 0 \end{pmatrix} \in \mathfrak{s}.$$

Denote the set of nilpotent elements in $\mathfrak{s}$ by $\mathcal{N}_\mathfrak{s}$. Then, by the above identification, $\beta \in M_{p,q}(\mathbb{C})$ belongs to $\mathcal{N}_\mathfrak{s}$ if and only if ${}^t\beta\beta$ is a nilpotent matrix, if and only if $\beta\, {}^t\beta$ is so.

Next we consider the symplectic group $G' = Sp(2n,\mathbb{R})$ of rank n. A maximal compact subgroup K' is isomorphic to the unitary group $U(n)$ of size n. To realize K' in a simple way, we define $Sp(2n,\mathbb{R})$ in a slightly different manner from the usual one. Namely, we put

$$\begin{aligned} G' &= U(n,n) \cap Sp(2n,\mathbb{C}) \\ &= \left\{ g \in GL(2n,\mathbb{C}) \mid {}^t\overline{g} 1_{n,n} g = 1_{n,n},\ {}^t g J g = J \right\}, \end{aligned}$$

where

$$1_{n,n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Then G' is isomorphic to $Sp(2n,\mathbb{R})$, and

$$K' = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} \mid k \in U(n) \right\} \subset G'$$

is a maximal compact subgroup. The corresponding Cartan decomposition is given by

$$\mathfrak{g}' = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^t H \end{pmatrix} \mid H \in \mathfrak{u}(n) \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C, D \in \mathrm{Sym}_n(\mathbb{C}) \right\} = \mathfrak{k}' \oplus \mathfrak{s}'.$$

We identify $\mathfrak{s}'$ with $\mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})$ via

$$\mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C}) \ni (C, D) \leftrightarrow \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \in \mathfrak{s}'.$$

Then (C, D) belongs to the nilpotent variety $\mathcal{N}_{\mathfrak{s}'}$ if and only if $C \cdot D$ is nilpotent, if and only if $D \cdot C$ is so.

Now we shall define the double fibration maps. Let $W = M_{p+q,n}(\mathbb{C})$ be the space of all the $(p+q) \times n$-matrices. We express a matrix Z in W as

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W; \qquad A \in M_{p,n}(\mathbb{C}), \quad B \in M_{q,n}(\mathbb{C}).$$

We define two maps φ and ψ by

$$\begin{aligned} \varphi :& W \ni Z \longmapsto A\, {}^t B \in M_{p,q}(\mathbb{C}) = \mathfrak{s}, \\ \psi :& W \ni Z \longmapsto ({}^t AA,\, {}^t BB) \in \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C}) = \mathfrak{s}'. \end{aligned}$$

Put

$$\begin{aligned} M_{\mathbb{C}} &= GL_p(\mathbb{C}) \times GL_q(\mathbb{C}) \supset O(p,\mathbb{C}) \times O(q,\mathbb{C}) = K_{\mathbb{C}}, \\ M'_{\mathbb{C}} &= GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \supset \Delta GL_n(\mathbb{C}) = K'_{\mathbb{C}}, \end{aligned}$$

and define $M_{\mathbb{C}} \times M'_{\mathbb{C}}$-action on W by

$$(m, m') \cdot Z = \begin{pmatrix} m_1 A\, {}^t m'_1 \\ m_2 B {m'_2}^{-1} \end{pmatrix} \qquad \text{for } Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W,$$

where

$$\begin{aligned} m &= (m_1, m_2) \in M_{\mathbb{C}} = GL_p(\mathbb{C}) \times GL_q(\mathbb{C}), \\ m' &= (m'_1, m'_2) \in M'_{\mathbb{C}} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}). \end{aligned}$$

We introduce $M_{\mathbb{C}}$-action on $\mathfrak{s}$ (resp. $M'_{\mathbb{C}}$-action on $\mathfrak{s}'$) so that $\varphi : W \to \mathfrak{s}$ is an $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant map (resp. $\psi : W \to \mathfrak{s}'$ is a $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-equivariant map). Note that the induced action is compatible with the adjoint $K_{\mathbb{C}}$-action on $\mathfrak{s}$ (resp. $K'_{\mathbb{C}}$-action on $\mathfrak{s}'$). As a $GL_n(\mathbb{C})$-module, the second component $\mathrm{Sym}_n(\mathbb{C})$ of $\mathfrak{s}'$ is regarded as the contragredient of the first component. By this reason, sometimes we will write $\mathfrak{s}' = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})^*$.

Our first observation is the following.

Lemma 1.1 *$\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ preserves nilpotent elements:*

$$\varphi(\psi^{-1}(\mathcal{N}_{\mathfrak{s}'})) \subset \mathcal{N}_{\mathfrak{s}}, \qquad \psi(\varphi^{-1}(\mathcal{N}_{\mathfrak{s}})) \subset \mathcal{N}_{\mathfrak{s}'}.$$

Proof. This is an easy consequence of direct calculations. Q.E.D.

Definition 1.2 Let $\mathcal{O}$ (resp. $\mathcal{O}'$) be a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{s}$ (resp. $K'_{\mathbb{C}}$-orbit in $\mathfrak{s}'$). If $\overline{\mathcal{O}} = \varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ holds, we say that $\mathcal{O}$ is the *theta lift* of $\mathcal{O}'$.

Note that $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ is an affine closed cone.

Proposition 1.3 *Assume that $2n < \min(p, q)$. Let $\mathcal{O}'_{[r,s]} = \mathcal{O}'_{\lambda_{r,s}} \subset \mathcal{N}_{\mathfrak{s}'}$ be a nilpotent $K'_{\mathbb{C}}$-orbit through*

$$\lambda_{r,s} = \left(\begin{array}{c|c} 0 & \begin{matrix} 1_r & \\ & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \\ & 1_s \end{matrix} & 0 \end{array} \right) \qquad (r + s \leq n).$$

Then there exists a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} \subset \mathcal{N}_{\mathfrak{s}}$ for which $\varphi(\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})) = \overline{\mathcal{O}}$ holds, i.e., the theta lift of $\mathcal{O}'_{[r,s]}$ exists. We denote $\mathcal{O} = \mathcal{O}_{[n;r,s]}$.

Remark 1.4 We allow $r = s = 0$, which means that $\mathcal{O}'_{[0,0]} = \{0\}$. Note that $\mathcal{O}'_{[r,s]}$ exhausts all the two-step nilpotent orbits in $\mathfrak{s}'$.

Proof. We will specify the nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} = \mathcal{O}_{[n;r,s]}$ in the end of the proof.

To prove the proposition, it suffices to prove that $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible. In fact, if it is irreducible, then $\varphi(\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}}))$ is an irreducible closed set, and is $K_{\mathbb{C}}$-stable in $\mathcal{N}_{\mathfrak{s}}$. Since $\mathcal{N}_{\mathfrak{s}}$ contains only a finite number of $K_{\mathbb{C}}$-orbits, it must be the closure of a single orbit.

Let us see that $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible. We call

$$\mathfrak{N}_{p,k} = \left\{ A \in M_{p,k}(\mathbb{C}) \mid {}^t A A = 0 \right\}$$

a *null cone* of size (p, k). It is known to be irreducible if $2k < p$. Thus, if we put

$$\begin{aligned} N_{r,s} = \Big\{ Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W \mid A = \left(\begin{array}{c|c} 1_r & 0 \\ \hline 0 & E \end{array} \right), B = \left(\begin{array}{c|c} 0 & 1_s \\ \hline F & 0 \end{array} \right), \\ \text{where } E \in \mathfrak{N}_{p-r,n-r} \text{ and } F \in \mathfrak{N}_{q-s,n-s} \Big\} \\ \simeq \mathfrak{N}_{p-r,n-r} \times \mathfrak{N}_{q-s,n-s}, \end{aligned}$$

then $N_{r,s}$ is irreducible and is contained in the fiber of $\lambda_{r,s}$. Moreover, under the condition that $2n < \min(p, q)$, it is easy to check that the exact fiber of $\lambda_{r,s}$ is given by

$$K_{\mathbb{C}}^{\circ} \cdot N_{r,s} = \psi^{-1}(\lambda_{r,s}),$$

where $K_{\mathbb{C}}^{\circ} \simeq SO(p,\mathbb{C}) \times SO(q,\mathbb{C})$ is the identity component of $K_{\mathbb{C}}$. Now we see that

$$(K_{\mathbb{C}}^{\circ} \times K'_{\mathbb{C}}) \cdot N_{r,s} = \psi^{-1}(\mathcal{O}'_{[r,s]}),$$

is irreducible, and hence $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible.

We can take the following matrix as a representative of a generic $K_{\mathbb{C}}^{\circ} \times K'_{\mathbb{C}}$-orbit in $N_{r,s}$.

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W;$$

$$A = \left(\begin{array}{c|c} 1_r & 0 \\ \hline 0 & 1_{n-r} \\ \hline 0 & 0 \\ \hline 0 & i1_{n-r} \end{array}\right) \in M_{p,n}(\mathbb{C}), \quad B = \left(\begin{array}{c|c} 1_{n-s} & 0 \\ \hline 0 & 1_s \\ \hline 0 & 0 \\ \hline i1_{n-s} & 0 \end{array}\right) \in M_{q,n}(\mathbb{C}). \tag{1.1}$$

By the above arguments, we know that the theta lift of $\mathcal{O}'_{[r,s]}$ should be exactly the $K_{\mathbb{C}}$-orbit through $\varphi(Z)$, where Z is given in (1.1). Q.E.D.

By the above proof, we conclude that the theta lift $\mathcal{O}_{[n;r,s]}$ of $\mathcal{O}'_{[r,s]}$ consists of at most three-step nilpotents. It is two-step nilpotent if and only if $r = s = 0$. Thus, we see that the theta lift of a k-step nilpotent orbit is a $(k+1)$-step nilpotent orbit.

2 Regular function ring of nilpotent orbits

In this section, we always assume that $2n < \min(p,q)$.

Let $\mathcal{O}'_{[r,s]} = \mathcal{O}'_{\lambda_{r,s}}$ $(r+s \leq n)$ be a nilpotent $K'_{\mathbb{C}}$-orbit in $\mathcal{N}_{\mathfrak{s}'}$ given in Proposition 1.3. We denote the corresponding theta lift by $\mathcal{O}_{[n;r,s]}$, which is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathcal{N}_{\mathfrak{s}}$.

We consider the case $s = 0$ in the following. Then we have

$$\mathcal{O}'_{[r,0]} = \{(C,0) \in \mathfrak{s}' \mid C \in \mathrm{Sym}_n(\mathbb{C}), \mathrm{rank}\, C = r\},$$

and it is known that $\overline{\mathcal{O}'_{[r,0]}}$ is the associated variety of an irreducible unitary highest weight representation of $Sp(2n,\mathbb{R})$ (or its metaplectic double cover). In particular, $\overline{\mathcal{O}'_{[n,0]}} \simeq \mathrm{Sym}_n(\mathbb{C})$ is the associated variety of a holomorphic discrete series representation of $Sp(2n,\mathbb{R})$.

Since $\mathcal{O}'_{[r,0]}$ is a $K'_{\mathbb{C}}$-orbit, the regular function ring $\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}]$ carries a natural $K'_{\mathbb{C}}$-module structure. Note that $K'_{\mathbb{C}} = GL_n(\mathbb{C})$. We denote by $\mathcal{P}_k$ all the partitions of length $\leq k$, i.e., $\mathcal{P}_k = \{\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0\}$.

Theorem 2.1 *The regular function ring $\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}]$ is decomposed as*

$$\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}] \simeq \sum_{\lambda \in \mathcal{P}_r}^{\oplus} \tau_{2\lambda}{}^* \qquad (\text{as a } GL_n(\mathbb{C})\text{-module}),$$

where τ_μ denotes an irreducible finite dimensional representation of $GL_n(\mathbb{C})$ with highest weight μ, and ${\tau_\mu}^*$ is its contragredient.

Proof. See [5], for example. Q.E.D.

Note that the fibration map

$$\psi : W = M_{p,n}(\mathbb{C}) \times M_{q,n}(\mathbb{C}) \longrightarrow \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{Sym}_n(\mathbb{C})^* = \mathfrak{s}'$$

is a product of two maps of the same kind,

$$\psi_p : M_{p,n} \ni A \longmapsto {}^tAA \in \mathrm{Sym}_n(\mathbb{C}) \quad \text{and}$$
$$\psi_q : M_{q,n} \ni B \longmapsto {}^tBB \in \mathrm{Sym}_n(\mathbb{C})^*.$$

Since $Sp(2n, \mathbb{R})/U(n)$ is a Hermitian symmetric space, $\mathfrak{s}'$ decomposes into two pieces of $K'_{\mathbb{C}}$-stable subspaces $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_-$, which we can identify with the decomposition $\mathfrak{s}' = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})^*$. Our orbit $\mathcal{O}'_{[r,0]}$ lives in $\mathfrak{s}'_+$ alone. Therefore, if we put $\Xi_{[r,0]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,0]}})$, it is decomposed as a product of closed affine cones

$$\Xi_{[r,0]} = \psi_p^{-1}(\overline{\mathcal{O}'_{[r,0]}}) \times \psi_q^{-1}(\{0\}) = \Xi_r^{(p)} \times \mathfrak{N}_{q,n},$$

where $\mathfrak{N}_{q,n}$ denotes the null cone given in the proof of Proposition 1.3, and

$$\begin{aligned} \Xi_r^{(p)} &= \psi_p^{-1}(\overline{\mathcal{O}'_{[r,0]}}) = \{A \in M_{p,n}(\mathbb{C}) \mid {}^tAA \in \overline{\mathcal{O}'_{[r,0]}}\} \\ &= \{A \in M_{p,n}(\mathbb{C}) \mid \mathrm{rank}\, {}^tAA \le r\}. \end{aligned}$$

Recall that

$$K'_{\mathbb{C}} = \Delta GL_n(\mathbb{C}) \subset GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) = M'_{\mathbb{C}}.$$

The following lemma is now clear.

Lemma 2.2 *The fiber $\Xi_{[r,0]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,0]}})$ is a product $\Xi_r^{(p)} \times \mathfrak{N}_{q,n}$, and hence it is $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-stable. The regular function ring breaks up into*

$$\mathbb{C}\big[\,\Xi_{[r,0]}\big] \simeq \mathbb{C}\big[\,\Xi_r^{(p)}\big] \boxtimes \mathbb{C}\big[\,\mathfrak{N}_{q,n}\big]$$

as an $(O(p, \mathbb{C}) \times GL_n(\mathbb{C})) \times (O(q, \mathbb{C}) \times GL_n(\mathbb{C}))$-module.

The regular function ring $\mathbb{C}\big[\,\mathfrak{N}_{q,n}\big]$ consists of precisely the $O(q, \mathbb{C})$-harmonic polynomials in $\mathbb{C}\big[\,M_{q,n}\big]$ (see [4], for example). As a consequence, it decomposes in a multiplicity-free manner,

$$\mathbb{C}\big[\,\mathfrak{N}_{q,n}\big] \simeq \sum_{\mu \in \mathcal{P}_n}^{\oplus} \sigma_\mu^{(q)} \boxtimes \tau_\mu \quad \text{(as an } O(q, \mathbb{C}) \times GL_n(\mathbb{C})\text{-module)}, \tag{2.1}$$

where $\sigma_\mu^{(q)}$ denotes an irreducible finite dimensional representation of $O(q, \mathbb{C})$ with highest weight μ. Let us decompose $\mathbb{C}\big[\,\Xi_r^{(p)}\big]$ as an $O(p, \mathbb{C}) \times GL_n(\mathbb{C})$-module,

$$\mathbb{C}\big[\,\Xi_r^{(p)}\big] \simeq \sum_{\lambda,\eta}^{\oplus} m(\lambda, \eta)\, \sigma_\eta^{(p)} \boxtimes {\tau_\lambda}^* \quad \text{(as an } O(p, \mathbb{C}) \times GL_n(\mathbb{C})\text{-module)}, \tag{2.2}$$

where $m(\lambda, \eta)$ denotes the multiplicity.

For $\lambda \in \mathcal{P}_n$, decompose an irreducible representation $\tau_\lambda^{(p)}$ of $GL_p(\mathbb{C})$ restricted to $O(p, \mathbb{C})$,

$$\tau_\lambda^{(p)}\big|_{O(p,\mathbb{C})} \simeq \sum_{\eta \in \mathcal{P}_n}^{\oplus} b_\eta^\lambda\, \sigma_\eta^{(p)}, \tag{2.3}$$

where b_η^λ denotes the branching coefficient. Note that η is also a partition of length $\leq n$.

Lemma 2.3 *The summation in* (2.2) *is taken over* $\lambda, \eta \in \mathcal{P}_n$*; and the multiplicity* $m(\lambda, \eta)$ *satisfies the following inequality,*

$$\delta_{\lambda,\eta} \leq m(\lambda, \eta) \leq b_\eta^\lambda, \tag{2.4}$$

where $\delta_{\lambda,\eta}$ *denotes Kronecker's delta. Moreover, we have a decomposition*

$$\mathbb{C}\big[\,\Xi_{[r,0]}\big] \simeq \sum_{\lambda,\mu,\eta \in \mathcal{P}_n}^{\oplus} m(\lambda, \eta)\, (\sigma_\eta^{(p)} \boxtimes \sigma_\mu^{(q)}) \boxtimes ({\tau_\lambda}^* \boxtimes \tau_\mu) \tag{2.5}$$

as an $(O(p, \mathbb{C}) \times O(q, \mathbb{C})) \times (GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))$*-module, where* $m(\lambda, \eta)$ *denotes the multiplicity given above.*

PROOF. Since $\Xi_r^{(p)}$ is a closed subvariety of $M_{p,n}$, $\mathbb{C}\big[\,\Xi_r^{(p)}\big]$ is a quotient of $\mathbb{C}\big[\,M_{p,n}\big]$. On the other hand, it is well known that $\mathbb{C}\big[\,M_{p,n}\big]$ decomposes as

$$\mathbb{C}\big[\,M_{p,n}\big] \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} {\tau_\lambda^{(p)}}^* \boxtimes {\tau_\lambda^{(n)}}^* \quad \text{(as a } GL_p(\mathbb{C}) \times GL_n(\mathbb{C})\text{-module)}.$$

Therefore, we have

$$\mathbb{C}\big[\,M_{p,n}\big] \simeq \sum_{\lambda,\eta \in \mathcal{P}_n}^{\oplus} b_\eta^\lambda\, \sigma_\eta^{(p)} \boxtimes {\tau_\lambda^{(n)}}^* \quad \text{(as an } O(p, \mathbb{C}) \times GL_n(\mathbb{C})\text{-module)}.$$

Now the second inequality in (2.4) is clear. The first inequality follows from the fact that $\mathfrak{N}_{p,n} \subset \Xi_r^{(p)}$ (cf. (2.1)). Q.E.D.

Theorem 2.4 *We assume that* $2n < \min(p, q)$*. Then the regular function ring of the theta lift* $\mathcal{O}_{[n;r,0]}$ *decomposes as*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;r,0]}}] \simeq \sum_{\lambda,\eta \in \mathcal{P}_n}^{\oplus} m(\lambda, \eta)\, \sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)} \tag{2.6}$$

as a $K_{\mathbb{C}} = O(p, \mathbb{C}) \times O(q, \mathbb{C})$*-module, where the multiplicity* $m(\lambda, \eta)$ *is given in* (2.2) *(cf. Lemma 2.3).*

We shall prove Theorem 2.4 in the next section.

Corollary 2.5 (1) *We have a multiplicity-free decomposition*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;0,0]}}] \simeq \sum_{\lambda \in \mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)} \boxtimes \sigma_\lambda^{(q)} \qquad \text{(cf. [4])}.$$

(2) *If we denote the branching coefficient of the restriction* $GL_p(\mathbb{C}) \downarrow O(p, \mathbb{C})$ *by* b_η^λ *(see* (2.3)*), the following decomposition holds.*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;n,0]}}] \simeq \sum_{\lambda,\eta \in \mathcal{P}_n}^{\oplus} b_\eta^\lambda\, \sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)}.$$

3 Harmonic polynomials and geometric quotient

In this section, we always assume that $2n < \min(p, q)$ as in the former section.

To prove Theorem 2.4, we study the induced algebra homomorphisms

$$\varphi^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[W], \quad \text{and} \quad \psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W].$$

Let us introduce a coordinate on $\mathfrak{s}'$. Take $(C, D) \in \mathfrak{s}'_+ \oplus \mathfrak{s}'_- = \mathfrak{s}'$, where $C = (C_{ij})$ and $D = (D_{ij})$ are symmetric matrices. We use $\{C_{ij} \mid 1 \le i \le j \le n\} \cup \{D_{ij} \mid 1 \le i \le j \le n\}$ as a coordinate on $\mathfrak{s}'$. Then ψ^* is given explicitly by

$$\psi^*(C_{ij}) = \sum_{k=1}^{p} A_{ki} A_{kj}; \quad \psi^*(D_{ij}) = \sum_{l=1}^{q} B_{li} B_{lj},$$

where $\{A_{ij} = Z_{ij} \mid 1 \le i \le p, 1 \le j \le n\} \cup \{B_{ij} = Z_{p+i,j} \mid 1 \le i \le q, 1 \le j \le n\}$ is considered as a system of coodinate functions on W which extracts the (i, j)-th element of $Z = \begin{pmatrix} A \\ B \end{pmatrix} \in M_{p+q,n}(\mathbb{C}) = W$. Note that the image of the coodinate functions via ψ^* is precisely the fundamental invariants for $K_{\mathbb{C}} = O(p, \mathbb{C}) \times O(q, \mathbb{C})$, which generate all the $K_{\mathbb{C}}$-invariants in $\mathbb{C}[W]$. Thus

$$\psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W]^{K_{\mathbb{C}}}$$

is surjective. Moreover, we have

Lemma 3.1 *Assume that $2n < \min(p,q)$. Then the map $\psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W]^{K_\mathbb{C}}$ is an isomorphism.*

Similarly, if we introduce a coordinate on $\mathfrak{s}$ by the (k,l)-th element of $X = (X_{kl}) \in M_{p,q}(\mathbb{C}) = \mathfrak{s}$, we see that

$$\varphi^*(X_{kl}) = \sum_{i=1}^{n} A_{ki} B_{li},$$

which is a fundamental invariant for $K'_\mathbb{C} = GL_n(\mathbb{C})$. Thus $\psi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]^{K'_\mathbb{C}}$ is surjective by the similar arguments as above. Let $\mathfrak{s}_{[n]} = \{X \in M_{p,q}(\mathbb{C}) \mid \operatorname{rank} X \leq n\}$ be the determinantal variety of rank n.

Lemma 3.2 *Assume that $2n < \min(p,q)$. Then the image of φ is precisely the determinantal variety: $\varphi(W) = \mathfrak{s}_{[n]}$. Thus the induced algebra homomorphism $\varphi^* : \mathbb{C}[\mathfrak{s}_{[n]}] \longrightarrow \mathbb{C}[W]^{K'_\mathbb{C}}$ is an isomorphism.*

The proof of the above two lemmas are almost immediate. We omit them.

Proposition 3.3 *Let $\mathcal{O}_{[n;r,s]}$ be the theta lift of $\mathcal{O}'_{[r,s]}$. Then $\overline{\mathcal{O}_{[n;r,s]}}$ is the geometric quotient of the fiber $\Xi_{[r,s]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ by $K'_\mathbb{C}$, i.e., $\overline{\mathcal{O}_{[n;r,s]}} = \Xi_{[r,s]}//K'_\mathbb{C}$. In particular, we have*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;r,s]}}] \simeq \mathbb{C}[\Xi_{[r,s]}]^{K'_\mathbb{C}}.$$

Proof. Let $J = \mathbf{I}(\Xi_{[r,s]})$ be the defining ideal of $\Xi_{[r,s]} \subset W$. Then, $I = (\varphi^*)^{-1}(J)$ is the defining ideal of $\overline{\mathcal{O}_{[n;r,s]}}$, since $\varphi(\Xi_{[r,s]}) = \overline{\mathcal{O}_{[n;r,s]}}$. Recall that $\varphi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]^{K'_\mathbb{C}}$ is surjective.

$$\begin{array}{ccc}
\mathbb{C}[\mathfrak{s}] & \xrightarrow{\varphi^* \,:\, \text{surjection}} & \mathbb{C}[W]^{K'_\mathbb{C}} \\
\text{projection}\downarrow & & \downarrow\text{projection} \\
\mathbb{C}[\overline{\mathcal{O}_{[n;r,s]}}] = \mathbb{C}[\mathfrak{s}]/I & \xrightarrow{\sim} & \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}}
\end{array}$$

Therefore, we get $\mathbb{C}[\mathfrak{s}]/I \simeq \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}}$. Note that $\mathbb{C}[\Xi_{[r,s]}]^{K'_\mathbb{C}} = (\mathbb{C}[W]/J)^{K'_\mathbb{C}} \simeq \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}}$. Thus, the proposition is proved. Q.E.D.

Let us consider the case where $s = 0$, and assume the decomposition (2.5). By the proposition above, we get

$$\begin{aligned}
\mathbb{C}[\overline{\mathcal{O}_{[n;r,0]}}] &\simeq \mathbb{C}[\Xi_{[r,0]}]^{K'_\mathbb{C}} \\
&= \sum_{\lambda,\mu,\eta \in \mathcal{P}_n}^{\oplus} m(\lambda,\eta)\,(\sigma_\eta^{(p)} \boxtimes \sigma_\mu^{(q)}) \boxtimes ({\tau_\lambda}^* \boxtimes \tau_\mu)^{\Delta GL_n(\mathbb{C})}.
\end{aligned}$$

By Shur's lemma, we have

$$({\tau_\lambda}^* \boxtimes \tau_\mu)^{\Delta GL_n(\mathbb{C})} \simeq \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \mathbb{C} & \text{if } \lambda = \mu. \end{cases}$$

Therefore, the above formula becomes

$$\mathbb{C}\big[\,\overline{\mathcal{O}_{[n;r,0]}}\,\big] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} m(\lambda,\eta)\,\sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)},$$

which finishes the proof of Theorem 2.4.

Finally, let us assume that $r = n$, and express the multiplicity $m(\lambda,\eta)$ by the Littlewood-Richardson coefficient $c_{\mu,\,\nu}^{\,\lambda}$, which is defined by the following formula

$$\tau_\mu \otimes \tau_\nu = \sum_{\lambda}^{\oplus} c_{\mu,\,\nu}^{\,\lambda} \tau_\lambda.$$

Proposition 3.4 *Let $\mathcal{O}_{[n;n,0]}$ be the theta lift of the open $K_{\mathbb{C}}$-orbit $\mathcal{O}'_{[n,0]}$ in $\mathfrak{s}'_+$. Then we get a $K_{\mathbb{C}}$-type decomposition*

$$\mathbb{C}\big[\,\overline{\mathcal{O}_{[n;n,0]}}\,\big] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} \left(\sum_{\mu\in\mathcal{P}_n} c_{\eta,\,2\mu}^{\,\lambda}\right) \sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)}.$$

Therefore, the multiplicity $m(\lambda,\eta)$ in Theorem 2.4 is given by

$$m(\lambda,\eta) = \sum_{\mu\in\mathcal{P}_n} c_{\eta,\,2\mu}^{\,\lambda},$$

for $r = n$.

Proof. In this case, we have $\Xi_n^{(p)} = M_{p,n}$. Let $\mathcal{H}$ be the space of all $O(p,\mathbb{C})$-harmonics in $\mathbb{C}\big[\,M_{p,n}\,\big]$. Then we have an isomorphism

$$\mathcal{H} \otimes \mathbb{C}\big[\,M_{p,n}\,\big]^{O(p,\mathbb{C})} \xrightarrow{\ \sim\ } \mathbb{C}\big[\,M_{p,n}\,\big]$$

given by the multiplication map. Thus we get

$$\mathbb{C}\big[\,\Xi_n^{(p)}\,\big] = \mathbb{C}\big[\,M_{p,n}\,\big] \simeq \mathcal{H} \otimes \mathbb{C}\big[\,M_{p,n}\,\big]^{O(p,\mathbb{C})} \simeq \mathcal{H} \otimes \mathbb{C}\big[\,\mathfrak{s}'_+\,\big].$$

From the following two decompositions,

$$\mathcal{H} \simeq \mathbb{C}\big[\,\mathfrak{N}_{p,n}\,\big] \simeq \sum_{\eta\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes {\tau_\eta}^* \quad \text{(as an } O(p,\mathbb{C}) \times GL_n(\mathbb{C})\text{-module)},$$

$$\mathbb{C}\big[\,\mathfrak{s}'_+\,\big] \simeq \sum_{\mu\in\mathcal{P}_n}^{\oplus} {\tau_{2\mu}}^* \quad \text{(as a } GL_n(\mathbb{C})\text{-module)},$$

we conclude that

$$\begin{aligned}
\mathbb{C}\big[\,\Xi_n^{(p)}\big] &\simeq \mathcal{H} \otimes \mathbb{C}\big[\,\mathfrak{s}'_+\big] \\
&\simeq \sum_{\eta,\mu\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes ({\tau_\eta}^* \otimes {\tau_{2\mu}}^*) \\
&\simeq \sum_{\eta,\mu\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes \sum_{\lambda\in\mathcal{P}_n}^{\oplus} c_{\eta,\,2\mu}^{\ \lambda} {\tau_\lambda}^* \\
&\simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} \left(\sum_{\mu\in\mathcal{P}_n} c_{\eta,\,2\mu}^{\ \lambda} \right) \sigma_\eta^{(p)} \boxtimes {\tau_\lambda}^*.
\end{aligned}$$

Q.E.D.

As an application of the above proposition, we get an interesting formula for the branching coefficient b_η^λ (see (2.3) for definition).

Corollary 3.5 *If* $2n < \min(p, q)$*, then we have*

$$b_\eta^\lambda = \sum_{\mu\in\mathcal{P}_n} c_{\eta,\,2\mu}^{\ \lambda} \qquad \text{for } \lambda, \eta \in \mathcal{P}_n.$$

Remark 3.6 The branching coefficient b_η^λ is naturally identified with the multiplicity of the K-type τ_λ in the holomorphic discrete series of $Sp(2n, \mathbb{R})$ with the minimal K-type τ_η. Thus, it does not depend on the particular value p, but only depends on $\lambda, \eta \in \mathcal{P}_n$.

PROOF. This follows from Corollary 2.5 (2). Q.E.D.

4 Further results and comments

Let us briefly discuss generalizations of the results above.

First, we note that we can develop the similar theory interchanging the role of the pair (G, G'), if $p + q \le n$ holds. So, if one of the pair is very small (i.e., if the pair is in the stable range), we can define the theta lifting from the smaller member of the pair to the larger one.

Almost all the arguments and results above are also valid for the other type I dual pairs with appropriate modifications. However, we must develop a new, unified language to describe them in general. For example, at present, we have to construct double fibration maps based on the case-by-case analysis. See the arguments in [6] for the pair $U(p, q) \times U(n, n)$.

Though the double fibration maps defined here might seem quite ad hoc, we have a natural interpretation for them, using the kernels and the images of nilpotent elements (cf. [7], [3]). Also there may be another interpretation by using moment maps. These interpretations will be useful for a general theory.

Our correspondence of nilpotent orbits is intimately related to the theta lifts of representations of $Sp(2n, \mathbb{R})$ to $O(p, q)$. The orbits $\mathcal{O}'_{[r,0]}$ treated in this note are associated to the unitary highest weight representations of $Sp(2n, \mathbb{R})$ (or its metaplectic double cover). In particular, $\mathcal{O}'_{[n,0]}$ corresponds to a holomorphic discrete series representation. Therefore, the theta lift $\mathcal{O}_{[n;n,0]}$ should be associated to the theta lift of a holomorphic discrete series. See [8] for the theta lift of the trivial representation, which is associated to the trivial orbit $\mathcal{O}'_{[0,0]} = \{0\}$.

Detailed discussions on the subjects commented above will appear elsewhere.

References

[1] R. Howe, Reciprocity laws in the theory of dual pairs. Progr. Math., 40, pp. 159–175, Birkhäuser Boston, Boston, Mass., 1983.

[2] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces. Amer. J. Math., 93 (1971), 753 – 809.

[3] H. Kraft and C. Procesi, Closures of conjugacy classes of matrices are normal. Invent. Math. 53 (1979), no. 3, 227–247.

[4] Kyo Nishiyama, Multiplicity-free actions and the geometry of nilpotent orbits. Preprint, 1999.

[5] Kyo Nishiyama, Hiroyuki Ochiai, and Kenji Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules (Hermitian symmetric case), Kyushu University Preprint Series in Mathematics, 1999-16.

[6] Kyo Nishiyama and Chen-Bo Zhu, Theta lifting of the trivial representation and the associated nilpotent orbit — the case of $U(p, q) \times U(n, n)$ —. In “Proceedings of Symposium on Representation Theory 1999” in Tateyama, Chiba, pp. 188 – 206.

[7] Takuya Ohta, The closures of nilpotent orbits in the classical symmetric pairs and their singularities. Tôhoku Math. J. (2) 43 (1991), no. 2, 161–211.

[8] C.-B. Zhu and J.-S. Huang, On certain small representations of indefinite orthogonal groups, Represent. Theory **1** (1997), 190–206 (electronic).

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