アフィン・グラスマン多様体のシューベルト・カルキュラス

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§1　10分でわかる(??)シューベルト・カルキュラス

『数え上げ幾何学講義』（池田岳）

§2　アフィン・グラスマン多様体とは？

T. Lam : Schubert Polynomials for the affine Grassmannian
JAMS 2008

§3　Symplectic 群の場合

講究録 2013年　Symplectic affine Grassmannian の同変 Schubert 類（池田岳）

中山．M. Shimozono

§1 シューベルト・カルキュラス

$g = (v_1, \cdots, v_n) \in GL_n(\mathbb{C})$

$gP \mapsto \langle v_1, \cdots, v_d \rangle$

$\in Gr(d, \mathbb{C}^n)$

グラスマン多様体

$$Gr(d, \mathbb{C}^n) = GL_n(\mathbb{C}) / P_d = G/P$$

$$GL_n(\mathbb{C}) \supset P_d = \left\{ \begin{array}{cc} & d \\ d & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \right\} = P : \text{parabolic}$$

Weyl group of $GL_n(\mathbb{C}) = S_n = W$

———— of $P_d = S_d \times S_{n-d} = W_P$

$$S_n / S_d \times S_{n-d} \cong W/W_P \cong W^P \text{ min reps}$$

$$= \{ w \in S_n \mid w(1) < \cdots < w(d),\ w(d+1) < \cdots < w(n) \}$$

シューベルト多様体

$$w \in W^P \Longrightarrow e_w = \mathrm{span}\{e_{w(1)}, \cdots, e_{w(d)}\} \in Gr(d, \mathbb{C}^n).$$

$T \subset GL_n(\mathbb{C})$
diagonal

$$Gr(d, \mathbb{C})^T = \{e_w \mid w \in W^P\}$$

Borel subgroup

$$B_- = \left\{ \left(\begin{smallmatrix} * & & \\ * & \ddots & \\ * & * & * \end{smallmatrix} \right) \right\}$$

$$w \in W^P \longrightarrow \Omega_w^\circ = B_- e_w \subset G/P$$

codim $|\lambda| = \ell(w)$
a cell

$$\Omega_w = \overline{\Omega_w^\circ}$$

Schubert variety

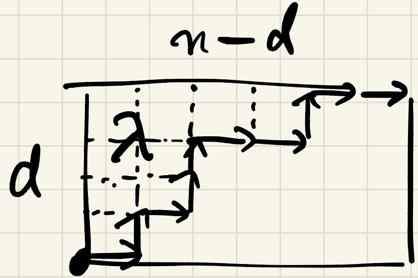
Example $w = id$, $e_{id} = \langle e_1, \cdots, e_d \rangle$

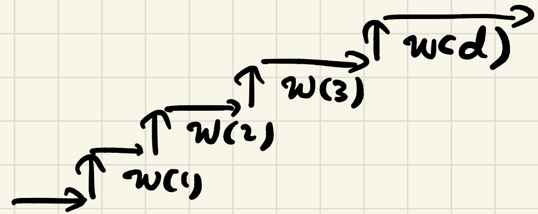
$$\Omega_{id}^\circ = B_- e_{id} \leftrightarrow \left\{ \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{array} \right) \right\} \cong \mathbb{C}^{d(n-d)}$$

the big cell.

ヤング図形 と シューベルト類

$w(1) < \cdots < w(d)$ から ヤング図形を作る

$\rightsquigarrow$ 



$$\lambda_1 = w(d) - d, \cdots, \lambda_{d-1} = w(2) - 2, \lambda_d = w(1) - 1.$$

$\lambda_1 \geq \cdots \geq \lambda_d \geq 0$

$\mathcal{P}_d^{n-d}$ 横ハバ 高さ

$$\lambda \in \mathcal{P}_d^{n-d} \rightsquigarrow \Omega_\lambda \rightsquigarrow \sigma_\lambda \in H^*(Gr(d, \mathbb{C}^n))$$

$\mathbb{Z}$-basis

"Schubert Calculus"

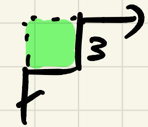
$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$$

$|\nu| = |\lambda| + |\mu|$ を満たす $\nu \in P_d^{n-d}$

$c_{\lambda\mu}^{\nu}$ は非負整数.

Example $Gr(2,\mathbb{C}^4)$ $\qquad w=(1\,3\,|\,2\,4)$

$$e_w = \left\{\begin{pmatrix}1&0\\0&0\\0&1\\0&0\end{pmatrix}\right\}, \qquad \Omega_w^0 \cong \left\{\begin{pmatrix}1&0\\ *&0\\0&1\\ *&*\end{pmatrix}\right\} = \mathbb{C}^3$$



$$\Omega_w = \{\, V \in Gr(2,\mathbb{C}^4) \mid \dim(F^2 \cap V) \geq 1 \,\}$$
$$\|$$
$$\Omega_\square$$

$\mathbb{P}^3 \supset l_0 \leftrightarrow F^2 = \left\{\begin{pmatrix}0\\0\\ *\\ *\end{pmatrix}\right\}$. $\qquad \mathbb{C}^4 = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = \{0\}$

$Gr(2,\mathbb{C}^4) \cong$ the lines ($\cong \mathbb{P}^1$) in $\mathbb{P}^3$.

$$\cup$$

$$\Omega_w = \{\, l \mid l \cap l_0 \neq \emptyset \,\}.$$

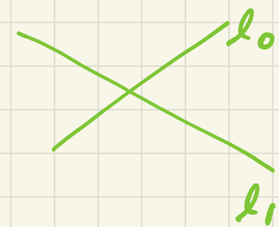
4 lines problem

$$l_1, \cdots, l_4 \in Gr(2, \mathbb{C}^4)$$

$$N = \#\{ l \in Gr(2, \mathbb{C}^4) \mid l \cap l_i \neq \emptyset \ (i = 1, 2, 3, 4) \}$$

$$\sigma_{\Box}^4 = N \cdot \sigma_{\boxplus}$$ を求める.

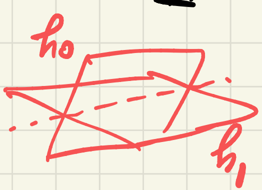
$$\Omega_{\boxplus} = \{ l_0 \}$$

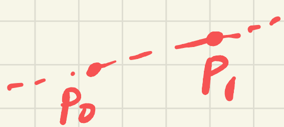


例 $\sigma_{\Box}^2 = \sigma_{\Box\Box} + \sigma_{\substack{\Box \\ \Box}}$

"$l \subset h_0$" "$p_0 \in l$"

$$\sigma_{\Box\Box}^2 = \sigma_{\substack{\Box \\ \Box}}^2 = \sigma_{\boxplus}, \quad \sigma_{\Box\Box} \cdot \sigma_{\substack{\Box \\ \Box}} = 0$$





$$\sigma_{\Box}^4 = (\sigma_{\Box\Box} + \sigma_{\substack{\Box \\ \Box}})^2 = 2\sigma_{\boxplus}.$$

Thm $\mathbb{Z}[z_1,\cdots,z_d]^{S_d} \twoheadrightarrow H^*(Gr(d,\mathbb{C}^n))$

surj. ring hom.

Schur pol.

$$s_\lambda(z_1,\cdots,z_d) \longmapsto \begin{cases} \sigma_\lambda & (\lambda\in P_d^{n-d}) \\ 0 & (\lambda\notin P_d^{n-d}) \end{cases}$$
$(\lambda \in P_d^\infty)$

Example $Gr(2,\mathbb{C}^4)$, $s_{\square} = z_1+z_2$.

$s_{\square\square} = z_1^2+z_1z_2+z_2^2$, $s_{\substack{\square\\\square}} = z_1z_2$

$s_{\square}^2 = z_1^2+2z_1z_2+z_1^2 = s_{\square\square} + s_{\substack{\square\\\square}}$

$s_{\square\square}\cdot s_{\substack{\square\\\square}} = \cancel{s_{\substack{\square\square\\\square}}} = 0$, $s_{\square\square}^2 = s_{\substack{\square\square\\\square\square}} + \cancel{s_{\substack{\square\square\square\\\square}}} + \cancel{s_{\square\square\square\square}}$,

$s_{\substack{\square\\\square}}^2 = s_{\substack{\square\square\\\square\square}}$

§2 アフィン・グラスマン多様体

Linear alg. group $G = SL_n$ etc (simply connected)

$$\widetilde{G} = G(\mathbb{K}),\quad \widetilde{P} = G(\mathbb{O})$$

$$\mathbb{O} = \mathbb{C}[[t]],\quad \mathbb{K} = \mathbb{C}((t)).$$

$$Gr_G = \widetilde{G}/\widetilde{P}.$$

coroot lattice

$$\underline{\text{Weyl group of}}\ \widetilde{G} = \widetilde{W} \cong Q^\vee \rtimes W$$

$$\widetilde{P} = W$$

$w \in \widetilde{W}$ に対して wW の unique 最短元を w^0 とする.

$$\widetilde{W}^0 = \{w^0 \mid w \in \widetilde{W}\}$$

$$\widetilde{W}/W \cong \widetilde{W}^0 \cong Q^\vee$$

min rep's

アフィン・グラスマニアンがなぜ面白いか？

$$H^*(\mathrm{Gr}_G) = \bigoplus_{w \in \widetilde{W}^0} \mathbb{Z}\,\sigma_w$$

$$\updownarrow \quad \text{Hopf dual}$$

$$H_*(\mathrm{Gr}_G) = \bigoplus_{w \in \widetilde{W}^0} \mathbb{Z}\,\xi_w$$

ring structure あり

Thm (Peterson 1997~, Lam–Shimozono 2010)

$$H_*(\mathrm{Gr}_G)_{loc} \cong QH^*(G/\cancel{B}\,P)_{loc}$$

J_P

シューベルト類も対応する

$B = (\quad) \subset GL_n$

$G/B \cong Fl_n(\mathbb{C})$

Gが古典群のとき

$\prod(1+x_i u) = \sum e_k u^k$
Morse

- $G = SL_n$: $\widetilde{W}^0 \cong \mathcal{P}_\infty^{n-1}$ $n-1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$

Lam-Schimozono $H^*(Gr_{SL_n}) \cong \mathbb{Z}[e_1, e_2, \cdots]/I_n^A$

$$\sigma_\lambda \longleftrightarrow F_\lambda^{(n)}$$

(affine Stanley function)

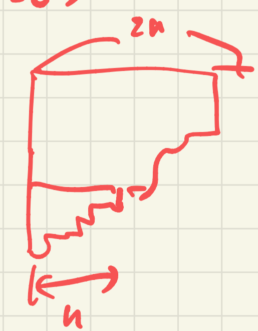
- $G = Sp_{2n}$: $\widetilde{W}^0 \cong \mathcal{P}_{C_n}^{(1)}$

$\lambda_1 \leq 2n$
$\lambda_i \leq n \Rightarrow \lambda_i > \lambda_{i+1}$

Lam-Schilling-Shimozono (Erikson-Erikson, Morse)

$$H^*(Gr_{Sp_{2n}}) \cong \mathbb{Z}[Q_1, Q_2, \cdots]/I_n^C$$

$$\sigma_\lambda \longleftrightarrow Q_\lambda^{(n)}$$



- $G = SO_n$: Pon

$T \subset G$ maximal torus

$T \cong (\mathbb{C}^\times)^r$

$H_T^*(Gr_G)$ algebra over

$S = H_T^*(pt) \cong \mathbb{Z}[t_1, \cdots, t_r]$

$$H_T^*(Gr_G) = \bigoplus_{w \in \widetilde{W}^0} S\, \sigma_w^T$$

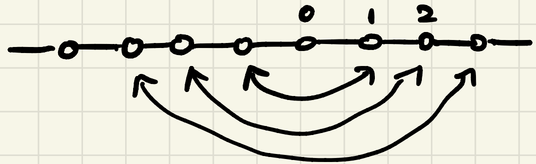
- $G = SL_n$ Lam–Shimozono

  double affine Stanley functions.

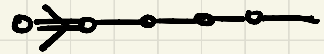
- $G = Sp_{2n}$ I–Nakayama–Shimozono

  Symmetric Function presentation

§3 Symplectic 群の場合 Folding Construction

$A_{\mathbb{Z}}$ 型 

からの folding

C_∞ 

$$
\begin{tikzcd}
H_T^*(Gr_{A_{\mathbb{Z}}}) \arrow[r] \arrow[d] & H_T^*(Gr_{A_{2n-1}}) \arrow[d] \\
H_T^*(Gr_{C_\infty}) \arrow[r] & H_T^*(Gr_{C_n})
\end{tikzcd}
$$

$A_{2n-1} \leftrightarrow GL_{2n}(\mathbb{C})$

$C_n \leftrightarrow Sp_{2n}(\mathbb{C})$

$H_T^*(Gr_{C_\infty}) = LG(\infty)$

Ivanov's factorial Q-functions (2003年ごろ)

$Q_k^{(r)}(x|t)$ を $\displaystyle\prod_{i=1}^{\infty}\left(\frac{1+x_i u}{1-x_i u}\right)\prod_{j=1}^{r-1}(1-t_j u)$ の u^k の係数とする.

$Q_k(x|t) := Q_k^{(k)}(x|t)$ は 1行の factorial Schur Q
(Ivanov)

2行 $k > \ell \geq 1$

$$Q_{k,\ell}(x|t) := Q_k^{(k)}(x|t)\,Q_\ell^{(\ell)}(x|t) + 2\sum_{j=1}(-1)^j Q_{k+j}^{(k)}(x|t)\,Q_{\ell-j}^{(k)}(x|t)$$

strict partition

$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_{2r} \geq 0)$ 　　λ の length $= \ell$ が odd のとき
$\ell + 1 = 2r$ として $\lambda_{2r} = 0$

$$Q_\lambda(x|t) := \mathrm{Pf}\left(Q_{\lambda_i,\lambda_j}(x|t)\right)_{1\leq i<j\leq 2r}$$

$\Gamma = \mathbb{Z}[Q_1, Q_2, \cdots]$

$Q_\lambda(x) := Q_\lambda(x|0)$ 　λ: strict は Γ の $\mathbb{Z}$-basis 　$= \displaystyle\bigoplus_{\lambda:\text{ strict}} \mathbb{Z} Q_\lambda$

$Q_\lambda(x|t)$ 　λ: strict は $\Gamma\otimes\mathbb{Z}[t]$ の $\mathbb{Z}[t]$-basis

$t = (t_1, t_2, \cdots)$

Thm (I 07) $\Gamma \otimes S \twoheadrightarrow H_T^*(LG(n))$ ← rank 2^n over S

$$Q_\lambda(x|t) \longrightarrow \begin{cases} \sigma_\lambda^T & (\lambda_1 \leq n) \\ 0 & (その他) \end{cases}$$

λ : strict

$n = \infty$ にできる.

$$\Gamma \otimes \mathbb{Z}[t] \xrightarrow{\sim} H_T^*(LG(\infty))$$

$$Q_\lambda(x|t) \longmapsto \sigma_\lambda^T$$

Thm (INS) $\exists$ surjective S-algebra hom

$$\Gamma \otimes_{\mathbb{Z}} S \longrightarrow H_T^*(Gr_{Sp_{2n}}), \quad H_T^*(pt) = S = \mathbb{Z}[t_1, \cdots, t_n]$$

Ivanov's factorial Q-function

- $\lambda \in \mathcal{P}_{C_n^{(1)}}$, $\lambda_1 \leqq n$ のとき $Q_\lambda(x|t) \mapsto \sigma_\lambda^T$

- $\lambda = \underbrace{\square\cdots\square}_{r}$ のとき $Q_{\square\cdots\square}(x|t^{(n)}) \mapsto \sigma^T_{\square\cdots\square}$ ← ring generators になっている

  $1 \leq r \leq 2n$ where $t^{(n)} = (t_1, \cdots, t_n, -t_n, \cdots, -t_1, t_1, \cdots\cdots)$

Kernel is generated by the deformed odd powersums ${}_2P_{2k+1}(x|t)$ $(k \geqq n)$.

Map の作り方

$$\mathrm{Gr}_G^{T} \cong \widetilde{W}^0$$
fixed pts

$$e_w \longleftrightarrow w$$

$i_w : e_w \hookrightarrow \mathrm{Gr}_G$ から

$$i_w^* : H_T^*(\mathrm{Gr}_G) \longrightarrow H_T^*(e_w) = S$$

$$H_T^*(\mathrm{Gr}_G) \longrightarrow \prod_{w \in \widetilde{W}^0} H_T^*(e_w) = \mathrm{Map}(\widetilde{W}^0, S)$$

$$\xi \longmapsto \left(\xi|_{e_w}\right)_{w \in \widetilde{W}^0}$$ は injective

Image の特徴付け (Lam-Schilling-Shimozono)

$\alpha \in \Delta_+$ に対し $\xi|_w - \xi|_{s_\alpha w} \in \alpha \cdot S$

$\alpha \in \Delta_+, d \geq 1$ に対し

$$\xi|_{(1-t_{\alpha^\vee})^d w} \in \alpha^d \cdot S$$

G の正ルート

$G = Sp_{2n}$ の場合 ([INS]) $(x_1, x_2, \cdots)$

各 $w \in \tilde{W}^0$ に対して $i_w^*: \Gamma \otimes S \to S$

を定義する $\tilde{W}^0 \cong Q^\vee = \bigoplus_{i=1}^n \mathbb{Z} t_i$, $S = \mathbb{Z}[t_1, \cdots, t_n]$

$w \longmapsto \sum_{i=1}^n m_i t_i$

$H^*(LG(\infty))$

$\wr\wr$ Pragacz

$\Gamma \ni f = f(x_1, x_2, \cdots)$

$\Gamma = \mathbb{Z}[Q_1(x), Q_2(x), \cdots]$

$= \bigoplus_{\lambda:\text{Strict}} \mathbb{Z} Q_\lambda(x)$

$\left(\begin{array}{l} m_i > 0 \text{ なら } t_i \text{ を } m_i \text{ 回} \\ m_i < 0 \quad -t_i \text{ を } \text{〃} \\ \text{それ以外 } 0 \text{ を 代入する} \end{array} \right.$

(Mac本 Chap III, §8)

例 $t_1 - 3t_2$ なら

$f(t_1, -t_2, -t_2, -t_2, 0, \cdots)$

$\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_1, p_3, p_5, \cdots]$