Orbit graphs of associated varieties

—joint work with

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Plan of talk

1. Motivation & Problems

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4. Induction of orbit graphs
   Define induction of orbit graphs, related to connectedness in codimension one
5. Associated varieties of HC-modules
   Show connected components of orbit graphs are associated varieties of certain HC-modules

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$K \subset G$: symmetric subgrp $\leftrightarrow$ involution θ

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$\mathfrak{g} := \mathrm{Lie}\,(G)_{\mathbb{C}} \rightsquigarrow \quad \underset{\theta}{\mathfrak{g}} = \underset{(+1)}{\mathfrak{k}} \oplus \underset{(-1)}{\mathfrak{s}}$: Cartan decomp

Nilpotent variety : $\mathcal{N}(\mathfrak{s}) = \mathfrak{s} \cap \mathcal{N}(\mathfrak{g}) \qquad G \curvearrowright \mathcal{N}(\mathfrak{g}),\ \ K \curvearrowright \mathcal{N}(\mathfrak{s})$

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Fact 1 : nilpotent variety

- $\#\mathcal{N}(\mathfrak{g})/\operatorname{Ad} G < \infty$: $\exists$ fin many # of G-orbits
- $\#\mathcal{N}(\mathfrak{s})/\operatorname{Ad} K < \infty$: $\exists$ fin many # of K-orbits

$\mathcal{O} \in \mathcal{N}(\mathfrak{g})/G$: nilpotent G-orbit $\rightsquigarrow$

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Fact 3: Associated Variety

- $\mathcal{AV}(\mathrm{Ann}\, X) = \overline{\mathcal{O}}$: irreducible
- $\mathcal{AV}(X) = \cup_{j=1}^r \overline{\mathbb{O}}_{i_j}$: irreducible decomp (reducible in general)

Vogan's theorem & Definition of orbit graph

Notation: $\partial\,\mathbb{O} = \overline{\mathbb{O}} \setminus \mathbb{O}$ codim $\partial\,\mathbb{O}$: codim of $\partial\,\mathbb{O}$ in $\overline{\mathbb{O}}$

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Definition ($\Gamma_K(\mathcal{O})$: orbit graph)

- Vertices : $\mathscr{V} = \{\mathbb{O}_i \mid 1 \leq i \leq \ell\}$: nilpotent K-orbits
- Edges : $\mathbb{O}_i - \mathbb{O}_j \iff$ codim $\partial\,\mathbb{O}_i \cap \partial\,\mathbb{O}_j = 1$ in $\overline{\mathbb{O}}_i$ (or $\overline{\mathbb{O}}_j$)

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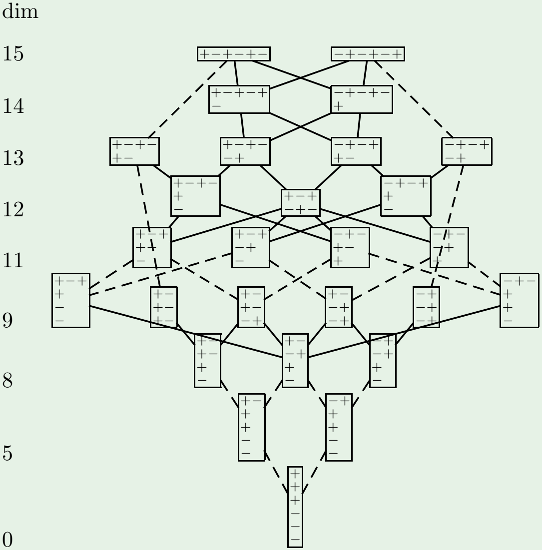
type AIII

We are concentrating on type AIII case:
$G = \mathrm{GL}_n \supset K = \mathrm{GL}_p \times \mathrm{GL}_q \;\; (n = p + q) \longleftrightarrow G_{\mathbb{R}} = \mathrm{U}(p, q)$

Closure ordering of nilpotent K-orbits for $(GL_6, GL_3 \times GL_3)$



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   $\exists$ irreducible HC-module X s.t. $Z = \mathcal{AV}^{\Gamma}(X)$?

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Some remarks:

- Notion of admissible orbits . . . Ohta(1991), Noël(2001)
- Orbits with different shapes can share codim 1 boundary
- Special piece? Singularities?

Orbit parametrization . . . Signed Young diagram

Young diagram (or partition) $\longleftrightarrow$ nilpotent G-orbits

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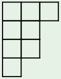
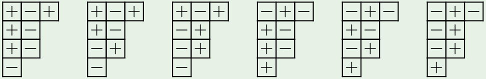
Examples of signed Young diagram $(p, q) = (4, 4)$

shape : $\lambda = (3, 2, 2, 1) =$  (Jordan blocks)

SYD : 

Express $\lambda \in \mathrm{YD}(n)$ in different manner

$$\lambda = (i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots, i_k, \ldots, i_k) \qquad i_1 > i_2 > \cdots > i_k > 0$$

$$= (i_1^{m(i_1)}, i_2^{m(i_2)}, \ldots, i_k^{m(i_k)}) \qquad m(i_j) > 0 \ \text{(multiplicity)}$$

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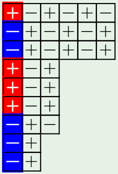
Example $\lambda = (6^3, 3^4, 2^2)$, $(p, q) = (18, 16)$



$$\longleftrightarrow \quad (m^+(6), m^+(3), m^+(2)) = (1, 3, 0)$$

Embedding

Define $\pi : \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda)) \simeq \mathrm{SYD}(\lambda; p, q) \to \mathbb{R}^k$ by

$$\pi(T) = (m^+(i_1), m^+(i_2), \ldots, m^+(i_k)) \in \mathbb{Z}^k_{\geq 0} \subset \mathbb{R}^k$$

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$m^+(i_r)$'s must satisfy 2 conditions

1. $0 \leq m^+(i_r) \leq m(i_r) \qquad (1 \leq r \leq k)$,
2. parity condition :
$$p - q = \sum_{i_r \text{ odd}} (m^+(i_r) - m^-(i_r)) = 2 \sum_{i_r \text{ odd}} m^+(i_r) - \sum_{i_r \text{ odd}} m(i_r).$$

Remark

Difference $m^+(i_r) - m^-(i_r)$ contributes only when row length i_r is odd (if even, the same number of $+$'s and $-$'s appear)

Recall the map:

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- *$\mathbb{Z}$-lattice points* $(x_i)_{1 \leq i \leq k}$ *in the hyper cube*

$$[0, m(i_1)] \times [0, m(i_2)] \times \cdots \times [0, m(i_k)]$$

satisfying the parity condition :

$$2 \sum_{i_r \text{ odd}} x_r = p - q + \sum_{i_r \text{ odd}} m(i_r)$$

Recall the map:

$$\pi : \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda)) \simeq \mathrm{SYD}(\lambda; p, q) \to \mathbb{R}^k$$

$$\pi(T) = (m^+(i_1), m^+(i_2), \dots, m^+(i_k)) \in \mathbb{Z}^k_{\geq 0}$$

Theorem

The map π so defined is a bijection between

- *nilpotent K-orbits* $\{\mathbb{O}_T \mid T \in \mathrm{SYD}(\lambda; p, q)\}$ *and*
- *$\mathbb{Z}$-lattice points* $(x_i)_{1 \leq i \leq k}$ *in the hyper cube*

$$[0, m(i_1)] \times [0, m(i_2)] \times \cdots \times [0, m(i_k)]$$

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Thus we are left to determine the edges of the orbit graph.

Theorem (Description of Edge)

Two vertices $\mathbb{O}_T, \mathbb{O}_{T'} \in \mathscr{V}(\Gamma_K(\mathcal{O}_\lambda))$ *are connected by edge*

$$\iff \quad \pi(T) - \pi(T') \in \{\pm(e_r - e_{r+1}) \mid 1 \leq r \leq k-1\} \cup \{\pm e_k\}$$

Here $\{e_r = (0, \ldots, 1, \ldots, 0) \mid 1 \leq r \leq k\}$ *denotes the std basis*

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Orbit graph for $(p, q) = (6, 6), \quad \lambda = (4, 3, 3, 1, 1)$

$\mathbb{Z}$-lattice pts (x_1, x_2, x_3) in

$$[0, 1] \times [0, 2] \times [0, 2]$$

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$$2(x_2 + x_3) = p - q + \#(\text{odd rows})$$
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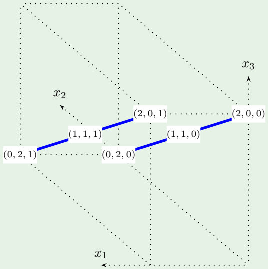
$\rightsquigarrow$ 2-conn components



Orbit graph for $(p, q) = (9, 9), \quad \lambda = (6, 4, 4, 2, 2)$

$\mathcal{O}_\lambda$: even nilpotent orbit

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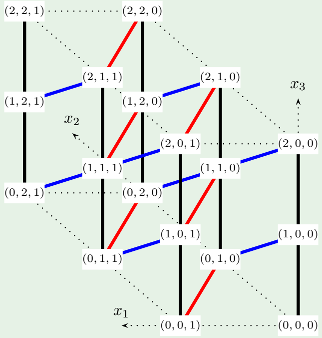

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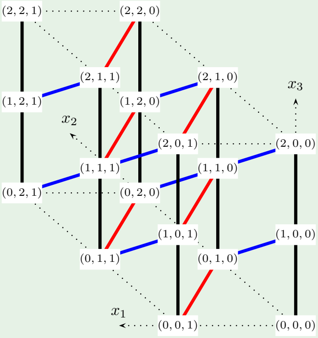

This always happens for even partitions...
$\rightsquigarrow$ $\exists$ generalization

Definition

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Consider the symmetric pair $(G, K) = (GL_n, GL_p \times GL_q)$ $(n = p + q)$

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Problem

How to control connected components?

Graph reduction

Given a Young diagram $\lambda \vdash n$

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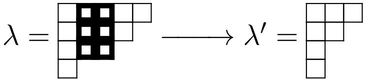
$\rightsquigarrow$ remove two successive columns of the same length from λ to get λ'



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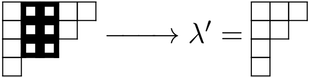
$\lambda =$  $\lambda' =$

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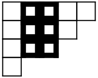
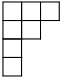
$\lambda =$  $\longrightarrow \lambda' =$ 

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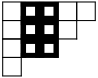
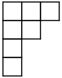
$\lambda =$  $\longrightarrow \lambda' =$ 

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This also works for signed Young diagram $T \in \mathrm{SYD}(\lambda; p, q)$

$\rightsquigarrow$ get a reduction map

$$\Phi : \mathrm{SYD}(\lambda; p, q) \to \mathrm{SYD}(\lambda'; p - h, q - h)$$

Graph Induction

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$(p, q) = (8, 7)$, $(p', q') = (3, 2)$; $\lambda' = (2^2, 1) \subset \lambda = (4^2, 3, 2^2)$

$$T' = \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline + \\ \cline{1-1}\end{array} \in \mathrm{SYD}(\lambda'; 3, 2) \rightsquigarrow \text{g-ind}(T') \subset \mathrm{SYD}(\lambda; 8, 7)$$

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Associated Graph

X : irred HC $(\mathfrak{g}, K)$-module

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Associated graph of X denoted as $\mathcal{AV}^{\Gamma}(X)$

$\cdots$ full subgraph in $\Gamma_K(\mathcal{O}_\lambda)$ with vertices $\{\mathbb{O}_i \mid 1 \leq i \leq r\}$
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Almost Theorem

If X is an irreducible HC-module,
associated graph $\mathcal{AV}^{\Gamma}(X)$ is a connected subgraph of $\Gamma_K(\mathcal{O}_\lambda)$

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   In fact, we can take $X =$ *(deg principal series) corr to parabolic subgrp whose Richardson orbit is* $\mathcal{O}$

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Example $G_{\mathbb{R}} = U(n, n)$

$P_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{C}) \ltimes N_{\mathbb{R}}$ $\quad \pi_\nu = \mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} |\det|^{\nu}$: deg principal series

$$\rightsquigarrow \mathcal{AV}^{\Gamma}(\pi_\nu) = \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline\end{array} \text{———} \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline - & + \\ \hline\end{array} \text{———} \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline - & + \\ \hline\end{array} \text{———} \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline - & + \\ \hline\end{array}$$

Maximally connected subgraph $\cdots$ connected compnts of $\Gamma_K(\mathcal{O})$

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$G = \mathrm{GL}_n \supset K = \mathrm{GL}_p \times \mathrm{GL}_q \ \ (n = p + q) \longleftrightarrow G_{\mathbb{R}} = \mathrm{U}(p, q)$

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- How to produce such HC-module X?

Maximally connected subgraph $\cdots$ connected compnts of $\Gamma_K(\mathcal{O})$

Keep considering type AIII
$G = \mathrm{GL}_n \supset K = \mathrm{GL}_p \times \mathrm{GL}_q \ \ (n = p + q) \longleftrightarrow G_{\mathbb{R}} = \mathrm{U}(p, q)$

Theorem (type AIII)

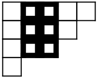
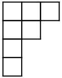
$\mathcal{O} \subset \mathfrak{g}$: *nilpotent* G*-orbit*
$\forall Z \subset \Gamma_K(\mathcal{O})$: *conn component*
$\quad \Longrightarrow \ \exists X$: *HC-module s.t.* $Z = \mathcal{AV}^{\Gamma}(X)$

- How to produce such HC-module X?

Key fact due to Barbasch-Vogan

$\forall \mathbb{O}$: K-orbit in $\mathfrak{s}$
$\exists X_{\mathbb{O}}$: irred derived functor module *s.t.* $\mathcal{AV}(X_{\mathbb{O}}) = \overline{\mathbb{O}}$

Recall λ and $\lambda' \rightsquigarrow$ remove two successive columns of the same length h from λ to get λ'

$\lambda =$  $\longrightarrow \lambda' =$ 

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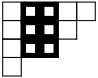
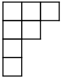
$\lambda =$  $\longrightarrow \lambda' =$ 

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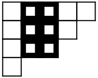
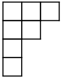
$\lambda =$  $\longrightarrow \lambda' =$ 

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$$\text{conn comp of } \Gamma_{K'}(\mathcal{O}_{\lambda'}) \to \text{conn comp of } \Gamma_K(\mathcal{O}_{\lambda})$$

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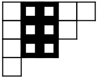
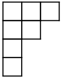
$\lambda =$  $\longrightarrow \lambda' =$ 

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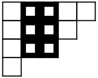
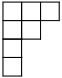
$\lambda =$  $\longrightarrow \lambda' =$ 

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In fact they match up!

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Theorem

For X' : HC $(\mathfrak{g}', K')$-module , put

$$X(\nu) := \mathrm{ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} X' \otimes |\det|^{\nu} \ : \ \underline{\textit{parabolic induction}}$$

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Starting from

- totally disconnected graph and
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we can thus construct

- X : HC-module with $\quad \mathcal{AV}^{\Gamma}(X) = (\text{conn comp of } \Gamma_K(\mathcal{O}))$

Thank you for your attention!!