Action of Hecke algebra on the double flag variety of type AIII

Kyo Nishiyama (西山 享)

AGU (青山学院大学理工)

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Introduction I

Among various properties of a double flag variety $\mathfrak{X} = K/B_K \times G/P$ of finite type, we focus on the followings:

- Geometry of orbits (dimensions, closure relations, conormal variety and moment maps, etc.)
- Combinatorics based on the graphs, Young tableaux, and RSK correspondence
- Representations of Hecke algebras, Weyl groups, and a kind of Springer-Steinberg theory

We report a definite results on the double flag variety of type AIII, or

$$\mathfrak{X} = \left(\mathscr{F}\!\ell(\mathbb{C}^p) \times \mathscr{F}\!\ell(\mathbb{C}^q)\right) \times \mathrm{Gr}_r(\mathbb{C}^{p+q}) \curvearrowleft K = \mathrm{GL}_p \times \mathrm{GL}_q$$

The talk is based on the (on-going) joint works with Lucas Fresse (IECL, Univ. Lorraine, France)

- Lucas Fresse and Kyo Nishiyama, *Action of Hecke algebra on the double flag variety of type AIII*, arXiv:2206.10476 [math.RT].
- ——, *A Generalization of Steinberg Theory and an Exotic Moment Map*, International Mathematics Research Notices (2020), rnaa080.
- ——, *Orbit embedding for double flag varieties and Steinberg map*, Contemp. Math. 768 (2021), 21–42.
- ——, *On generalized Steinberg theory for type AIII*, 2021, arXiv:2103.08460.

Double flag varieties and K-orbits I

Consider:

- G : connected reductive algebraic group
- K : symmetric subgroup
- $P \subset G$, $Q \subset K$: parabolic subgroups (psg)

The double flag variety $\mathfrak{X}$ is introduced in (N-Ochiai 2011 [8])

$$\mathfrak{X} = G/P \times K/Q \curvearrowleft K$$

Usually $\#\mathfrak{X}/K = \infty$, but $\exists$ interesting cases where $\#\mathfrak{X}/K < \infty$, called finite type

even $\exists$ classification (He-N-Ochiai-Oshima 2013 [4]) of $\mathfrak{X}$ of finite type

(when $P = B_G$ or $Q = B_K$, i.e., one of them is a Borel subgrp)

Note that $\mathfrak{X}/K \simeq Q\backslash G/P$ (preserving closure relations)

Double flag variety of type AIII I

Take $\mathbb{C}$ as a base field (for convenience)

In this talk, we will concentrate on the case of symmetric sp of type AIII:

- $G = \mathrm{GL}_n$: general linear group
- $K = \mathrm{GL}_p \times \mathrm{GL}_q$: block diagnal subgrp of G $\quad (p + q = n)$
- $P = P_{(r,n-r)}$: max psg in G (with 2 diag blocks of size r & $n - r$)
- $Q = B_K = B_p \times B_q$: Borel subgrp in K

So that
$$\begin{aligned}\mathfrak{X} &= \mathrm{GL}_n/P_{(r,n-r)} \times \mathrm{GL}_p/B_p \times \mathrm{GL}_q/B_q \\ &\simeq \mathrm{Gr}_r(V) \times \mathscr{F}\!\ell(V^+) \times \mathscr{F}\!\ell(V^-),\end{aligned}$$
where

- $V = V^+ \oplus V^-$ $(V^+ = \mathbb{C}^p,\ V^- = \mathbb{C}^q)$ polar decomposition
- $\mathrm{Gr}_r(V)$: Grassmannian of r-dim subsp's of V
- $\mathscr{F}\!\ell(V^\pm)$: complete flag varieties

Double flag variety of type AIII II

Lemma

$\#\mathfrak{X}/K < \infty$, *i.e.,* $\mathfrak{X}$ *is of finite type*

Write $X = \mathrm{Gr}_r(V) = G/P_{(r,n-r)}$ $\rightsquigarrow$ $K \curvearrowright X$: spherical (i.e., $\#X/B_K < \infty$)

Lemma

$$\begin{array}{ccc} \mathfrak{X}/K & \xrightarrow{\simeq} & X/B_K \\ \Uplus & & \Uplus \\ K \cdot ([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-) & \longmapsto & B_K \cdot [\tau] \end{array}$$

where $[\tau] \in \mathrm{Gr}_r(V)$, $\mathcal{F}_0^{\pm}$*: standard flags stabilized by* B_p *or* B_q

We will often identify $\mathfrak{X}/K \simeq X/B_K$

Let us describe what is the representative $\{\tau\}$

Description of K orbits on $\mathfrak{X}$ I

Partial permutation: $\tau_1 \in \mathfrak{T}_{p,r} \subset \mathrm{M}_{p,r}$ with entries of 0 or 1, in which $\#1 = 0$ or 1 in $\forall$rows and columns

$$\mathfrak{T} = \mathfrak{T}_{(p,q),r} := \left\{ \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathfrak{T}_{p,r} \times \mathfrak{T}_{q,r} \,\middle|\, \operatorname{rank} \tau = r \right\} \subset \mathrm{M}_{p+q,r}$$

pairs of partial permutations of full rank

$$\mathfrak{T} \curvearrowleft \mathfrak{S}_r \qquad \rightsquigarrow \overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r \text{ : quotient by the symmetric group action}$$

Denote $[\tau] := \operatorname{Im} \tau \in \mathrm{Gr}_r(V)$: r-dim subsp gented by column vectors of τ

Theorem

$\mathfrak{T} \ni \tau \mapsto [\tau] \in \mathrm{Gr}_r(V)$ *factors through to*

$$\overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r \xrightarrow{\simeq} X/B_K \simeq \mathfrak{X}/K$$

so that we get $\overline{\mathfrak{T}} \simeq \mathfrak{X}/K$

$\exists$ convenient presentation by graphs...

Graphs I

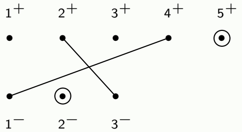
Denote $\tau \in \overline{\mathfrak{T}}$ by a graph $\Gamma(\tau)$:

- signed Vertices: positive vertices $\mathcal{V}_p^+ = \{1^+, \dots, p^+\}$ & negative vertices $\mathcal{V}_q^- = \{1^-, \dots, q^-\}$
- Edges between $i^+ \in \mathcal{V}_p^+$ and $j^- \in \mathcal{V}_q^-$ if τ contains two 1's at the positions i^+ & j^- in the same column
- Marking at the vertex i^+ or j^-, with only one 1 at i^+ or j^- in a column
- $\#(\text{Edges}) + \#(\text{Marked points}) = r$

Example

$(p, q) = (5, 3)$ and $r = 4$,

$$\tau = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right) = \left(\frac{\boldsymbol{e}_2 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_5 \quad 0}{\boldsymbol{e}_3 \quad \boldsymbol{e}_1 \quad 0 \quad \boldsymbol{e}_2}\right) \rightsquigarrow \Gamma(\tau) =$$



$\mathcal{G}((p, q), r)$: the set of graphs with vertices $\mathcal{V}_p^+ \cup \mathcal{V}_q^-$ & exactly r edges & marked points

Graphs II

Lemma

The graphs classify K orbits in $\mathfrak{X}$

$$\begin{array}{ccccc} \mathfrak{X}/K \simeq X/B_K & \xleftarrow{\ \simeq\ } & \overline{\mathfrak{T}} & \xrightarrow{\ \simeq\ } & \mathcal{G}((p,q),r) \\ \Cup & & \Cup & & \Cup \\ B_K \cdot [\tau] & \longleftarrow\!\dashv & \tau & \longmapsto & \Gamma(\tau) \end{array}$$

Orbital invariants: $a^{\pm}(\tau)$, $b(\tau)$, $c(\tau)$ & $R(\tau) = (r_{i,j}(\tau))$ I

For the graph $\Gamma(\tau)$ we define:

- degree of vertices: $\deg i^{\pm} := 0, 1, 2$ (depending on NO [edges/marks], edges, marked resp.)
- $a^{\pm}(\tau) := \#\{(i^{\pm}, j^{\pm}) \mid i < j \;\&\; \deg(i^{\pm}) < \deg(j^{\pm})\}$
- $b(\tau) := \#\{\text{edges}\}$
- $c(\tau) := \#\{\text{crossings}\}$
- $r_{i,j}(\tau) :=$ #Edges + #Marks within vertices among $1^+ \leqslant k^+ \leqslant i^+$ & $1^- \leqslant \ell^- \leqslant j^-$

$$R(\tau) := \big(r_{i,j}(\tau)\big)_{0 \leqslant i \leqslant p,\, 0 \leqslant j \leqslant q} \in \mathrm{M}_{p+1,q+1} : \text{the rank matrix}$$

- decomposition $\mathcal{V}_p^+ = \{1, \ldots, p\} = I \sqcup L \sqcup L'$:

  I (resp. L, resp. L') denotes the set of elements $i \in \{1, \ldots, p\}$ such that i^+ is a vertex of $\Gamma(\tau)$ of degree 1 (resp. 2, resp. 0)

- Similar decomp $\mathcal{V}_q^- = \{1, \ldots, q\} = J \sqcup M \sqcup M'$:

  J (resp. M, resp. M') consists of the elements j such that j^- has degree 1 (resp. 2, resp. 0)

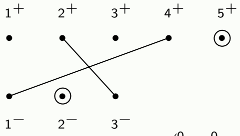
- Let $\sigma : J \to I$ be the bijection defined by $\sigma(j) = i$ if (i^+, j^-) is an edge in $\Gamma(\tau)$.

Orbital invariants: $a^{\pm}(\tau)$, $b(\tau)$, $c(\tau)$ & $R(\tau) = (r_{i,j}(\tau))$ II

Example

Let τ be as in (4):

$$\tau = \binom{\tau_1}{\tau_2} = \left(\begin{array}{cccc} \boldsymbol{e}_2 & \boldsymbol{e}_4 & \boldsymbol{e}_5 & 0 \\ \hline \boldsymbol{e}_3 & \boldsymbol{e}_1 & 0 & \boldsymbol{e}_2 \end{array} \right) \rightsquigarrow \Gamma(\tau) =$$



then,

$$a^{+}(\tau) = 7, \quad a^{-}(\tau) = 1, \quad b(\tau) = 2, \quad c(\tau) = 1, \quad R(\tau) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$I = \{2, 4\}, \quad L = \{5\}, \quad L' = \{1, 3\},$$

$$J = \{1, 3\}, \quad M = \{2\}, \quad M' = \varnothing, \quad \sigma = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in \mathrm{Bij}(J, I).$$

Dimensions and closure relations of Orbits I

Recall a based point $([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-)$ in $\mathfrak{X} = \mathrm{Gr}_r(V) \times \mathscr{F}\!\ell(V^+) \times \mathscr{F}\!\ell(V^-)$.

Theorem

Denote a K orbit in $\mathfrak{X}$ by $\mathbb{O}_\tau := K \cdot ([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-)$

1. $\dim \mathbb{O}_\tau = \dfrac{p(p-1)}{2} + \dfrac{q(q-1)}{2} + a^+(\tau) + a^-(\tau) + \dfrac{b(\tau)(b(\tau)+1)}{2} + c(\tau).$
2. $\mathbb{O}_\tau = \{(W, \mathcal{F}^+, \mathcal{F}^-) \mid \dim W \cap (\mathcal{F}_i^+ + \mathcal{F}_j^-) = r_{i,j}(\tau) \quad . \quad \forall (i,j) \in \{0, \dots, p\} \times \{0, \dots, q\}\}$
3. $\overline{\mathbb{O}_\tau} \subset \overline{\mathbb{O}_{\tau'}} \iff r_{i,j}(\tau) \geqslant r_{i,j}(\tau') \;\; \forall (i,j) \in \{0, \dots, p\} \times \{0, \dots, q\}.$

We can describe the cover relation of the closure of orbits, which is not given today. See our recent preprint Fresse-N `arXiv:2103.08460` [2] for details.

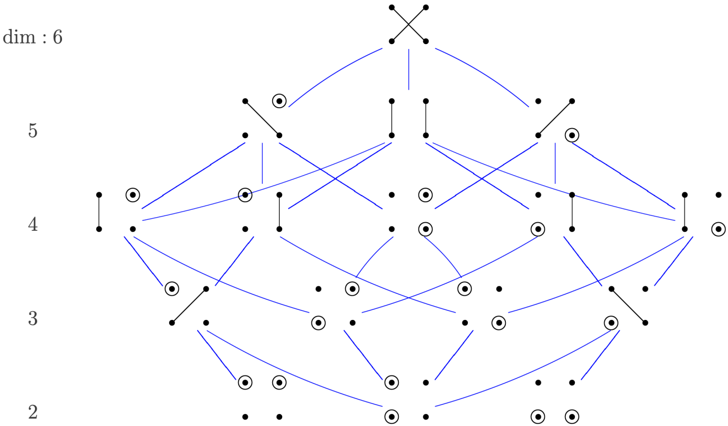
However, we have

Corollary

$\mathbb{O}_{\tau'}$ *covers* $\mathbb{O}_\tau \implies \dim \mathbb{O}_{\tau'} = \dim \mathbb{O}_\tau + 1$

Dimensions and closure relations of Orbits II

Figure: Closure relations of K orbits for $p = q = r = 2$



The number of orbits I

Let (k, s, t) be nonnegative integers s.t.

$$p \geqslant k + s, \quad q \geqslant k + t, \quad r = k + s + t$$

$$\text{Put} \qquad s' = p - k - s \quad \& \quad t' = q - k - t$$

Consider a subgroup $H_{k,s,t} \subset \mathfrak{S}_p \times \mathfrak{S}_q$:

$$\begin{aligned} H_{k,s,t} &= \{(a_1, a_2, a_3; a_1, b_2, b_3) \in (\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'}) \times (\mathfrak{S}_k \times \mathfrak{S}_t \times \mathfrak{S}_{t'})\} \\ &\cong \Delta\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'} \times \mathfrak{S}_t \times \mathfrak{S}_{t'}, \end{aligned}$$

where $\Delta\mathfrak{S}_k \subset \mathfrak{S}_k^2$ stands for the diagonal subgroup

Theorem

The total number of K-orbits in $\mathfrak{X}$ is given by

$$\#\mathfrak{X}/K = \sum_{(k,s,t)} \dim \operatorname{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p \times \mathfrak{S}_q} \mathbf{1} = \sum_{(k,s,t)} \binom{p}{k, s, s'} \binom{q}{k, t, t'} k!,$$

where the sums are running over triples (k, s, t) given above.

Hecke algebra action I

$\exists$ standard way to define Hecke algebra action of $\mathscr{H} = \mathscr{H}(K, B_K)$ on the DFV $\mathfrak{X}$ by the convolution product:

$$\begin{array}{ccc} & K/B_K \times K/B_K \times G/P & \\ {\scriptstyle p_{12}} \swarrow & & \searrow {\scriptstyle p_{23}} \\ K/B_K \times K/B_K & & K/B_K \times G/P = \mathfrak{X} \end{array}$$

However, we prefer a simpler picture

$$\begin{array}{ccc} & K \times_{B_K} G/P & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ K/B_K & & G/P = X \end{array}$$

More generally, if X is a spherical K-variety, Hecke algebra actions are considered by Mars-Springer [7] and Knop [6].

Calculation over finite fields I

To get an explicit formula of the action of $\mathscr{H}$, we follow the old recipe of Iwahori [5]:

Let us consider everything over a finite field $\mathbb{F}$ (of characteristic p, say[1]).

Summary of the notations (ignore the first column):

G	K	$\mathrm{GL}_p(\mathbb{F}) \times \mathrm{GL}_q(\mathbb{F})$
H	B_K	$B_p(\mathbb{F}) \times B_q(\mathbb{F})$
W	W_K	$S_p \times S_q$
X	G/P	$\mathrm{GL}_n(\mathbb{F})/P_{(r,n-r)}(\mathbb{F}) \simeq \mathrm{Gr}_r(\mathbb{F}^n)$
T	$\mathfrak{T} = B_K \backslash G/P$	$\mathfrak{T}_{(p,q;r)}$

- $s_i = (i, i+1)$: simple reflection (a transposition in W_K), &
  $T_i = T_{s_i}$: corresp generator in $\mathscr{H}$
- Recall pairs of partial permutations $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r$ of full rank r
  τ is identified with its image $[\tau] \in X = \mathrm{Gr}_r(\mathbb{F}^q)$ (thus we often omit [ ] below)
- Let $\mathbb{O}_\tau = B_K \cdot \tau$ be a B_K-orbit of the Grassmannian X
  ξ_τ denotes the characteristic function of the orbit $\mathbb{O}_\tau$

We are interested in $T_i * \xi_\tau$ for $\tau \in \mathfrak{T}$.

[1] This conflicts with our notation (p, q, r).

Orbit multiplications I

As in the above, $\tau \in \mathfrak{T}$ is often identified with a graph with two kinds of vertices $\mathcal{V}_p^+$ and $\mathcal{V}_q^-$ of p and q elements respectively which are equipped with several markings and edges.

Lemma

A double coset $B_K s_i B_K$ generates at most two B_K orbits on the Grassmannian $X = \mathrm{Gr}_r(\mathbb{F}^n)$. Namely we have

$$B_K s_i B_K \cdot \tau = \begin{cases} B_K s_i \tau = B_K \tau & \text{if } s_i\tau = \tau \qquad \text{case (I)} \\ B_K s_i \tau \cup B_K \tau & \text{if } s_i\tau \neq \tau \text{ and } \tau \text{ is in case (II)} \\ B_K s_i \tau & \text{if } s_i\tau \neq \tau \text{ and } \tau \text{ is in case (III)} \end{cases}$$

where

- case (I): $i, i+1$ *are both of degree 0 (isolated) or both of degree 2 (marked)*
- case (II): $\deg_\tau(i) < \deg_\tau(i+1)$ *or* $i, i+1$ *are end points of edges with crossing*
- case (III): $\deg_\tau(i) > \deg_\tau(i+1)$ *or* $i, i+1$ *are end points of edges without crossing*

Explicit action of Hecke algebra on the double flag variety I

By Lemma 10, we get

$$T_i * \xi_\tau = \alpha\, \xi_\tau + \beta\, \xi_{s_i \tau}$$

for some coefficients $\alpha, \beta \in \mathbb{Q}$ (one of which might be zero).
Explicit calculations of the convolution product tells us

$$\alpha = \frac{\#\big(K_\tau B_K \cap B_K s_i B_K\big)}{\# B_K}$$

$$\beta = \frac{\#\big(s_i K_\tau B_K \cap B_K s_i B_K\big)}{\# B_K}$$

where $K_\tau = \mathrm{Stab}_K([\tau]) = K \cap P_{[\tau]}$ is the stabilizer of $[\tau] \in X$.

Conclusion I

Theorem

Denote $q = \#\mathbb{F}$, the number of elements in $\mathbb{F}$, then the action of the generators in $\mathscr{H} = \mathscr{H}(K, B_K)$ is given by

$$T_i * \xi_\tau = \begin{cases} q\,\xi_\tau & (s_i\tau = \tau) \qquad \text{Case (I)} \\ (q-1)\,\xi_\tau + q\,\xi_{s_i\tau} & (s_i\tau \neq \tau) \qquad \text{Case (II)} \\ \xi_{s_i\tau} & (s_i\tau \neq \tau) \qquad \text{Case (III)} \end{cases}$$

they satisfy the Hecke algebra relations:

$$(T_s + 1)(T_s - q) = 0 \tag{3.1}$$

$$T_{ww'} = T_w T_{w'} \qquad \text{if } \ell(ww') = \ell(w) + \ell(w') \tag{3.2}$$

If we specialize $q = 1$, then the Hecke alg rep goes down to that of the Weyl group

$W_K = \mathfrak{S}_p \times \mathfrak{S}_q$. It's just a permutation on the vertices of the graphs.
Thus we get

Conclusion II

Corollary

The representation of $W_K = \mathfrak{S}_p \times \mathfrak{S}_q$ *on* $\mathbb{C}[\mathfrak{X}/K]$ *is isomorphic to*

$$\bigoplus_{(k,s,t)} \mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p \times \mathfrak{S}_q} \mathbf{1}$$

where (k, s, t) *moves over*

$$p \geqslant k + s, \quad q \geqslant k + t, \quad r = k + s + t$$

and

$$H_{k,s,t} \simeq \Delta\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'} \times \mathfrak{S}_t \times \mathfrak{S}_{t'},$$

with $s' = p - (k + s)$ *&* $t' = q - (k + t)$*.*

Generalized/Exotic Steinberg theory I

We can describe the basis of the representation by Young tableaux through generalized/exotic Steinberg map

(Fresse-N, 2020 IMRN and Contemp. Math. [1, 3]).

Recall the double flag variety (in a broader context)

$$\mathfrak{X} = \mathscr{P}_G \times \mathscr{B}_K, \qquad \mathscr{P}_G = G/P, \ \mathscr{B}_K = K/B_K$$

As usual, we identify $\mathscr{P}_G = G/P$ with the set of parabolic subalgs $\mathfrak{p}_1 \overset{conj}{\sim} \mathfrak{p} = \mathrm{Lie}\,(P)$.

Then the cotangent bundle over $\mathscr{P}_G$:

$$T^*\mathscr{P}_G = \{(\mathfrak{p}_1, x) \mid \mathfrak{p}_1 \in \mathscr{P}_G, \ x \in \mathfrak{u}_{\mathfrak{p}_1}\} \simeq G \times_P \mathfrak{u}_{\mathfrak{p}},$$

where $\mathfrak{u}_{\mathfrak{p}}$ = nilradical$(\mathfrak{p})$

The moment map:

$$\begin{array}{cccl} \mu_{\mathscr{P}_G} : & T^*\mathscr{P}_G & \longrightarrow & \mathcal{N}_{\mathfrak{g}} \quad = \text{(nullcone)} \quad \text{(2nd proj)} \\ & \Uplus & & \Uplus \\ & (\mathfrak{p}_1, x) & \longmapsto & x \end{array}$$

wrt a standard symplectic structure on $T^*\mathscr{P}_G$.

Generalized/Exotic Steinberg theory II

Similarly, we have the moment map

$$\mu_{\mathscr{B}_K} : T^*\mathscr{B}_K = \{(\mathfrak{q}_1, y) \mid \mathfrak{q}_1 \in \mathscr{B}_K,\ y \in \mathfrak{u}_{\mathfrak{q}_1}\} \to \mathcal{N}_{\mathfrak{k}}, \qquad \mu_{\mathscr{B}_K}(\mathfrak{q}_1, y) = y$$

Definition

$\mathcal{Y} := T^*\mathscr{P}_G \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathscr{B}_K$: fiber product over nilpotent var $\mathcal{N}_{\mathfrak{k}}$:

$$\begin{array}{ccc}
\mathcal{Y} = T^*\mathscr{P}_G \times_{\mathcal{N}_{\mathfrak{k}}} T^*\mathscr{B}_K & \xrightarrow{\ p_1\ } & T^*\mathscr{P}_G \ni (\mathfrak{p}_1, x) \\
 & & \downarrow \mu_{\mathscr{P}_G} \\
\downarrow p_2 & \searrow \varphi^{\theta} & \mathcal{N}_{\mathfrak{g}} \ni x \\
 & & \downarrow \mathrm{pr}_{\mathfrak{k}} \\
(\mathfrak{q}_1, y) \in T^*\mathscr{B}_K & \xrightarrow{\ -\mu_{\mathscr{B}_K}\ } & \mathcal{N}_{\mathfrak{k}} \ni -y = x^{\theta}
\end{array}$$

We call $\mathcal{Y} = \mathcal{Y}_{\mathfrak{X}}$ the conormal variety for the double flag variety $\mathfrak{X}$.

Generalized/Exotic Steinberg theory III

Fact

1. *Let* $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \to \mathfrak{k}$ *be the moment map on* $T^*\mathfrak{X}$.
   Then $\mathcal{Y} \simeq \mu_{\mathfrak{X}}^{-1}(0)$ *is the null fiber.*
2. $\mathcal{Y}$ *is a disjoint union of the conormal bundles:* $\mathcal{Y} = \coprod_{\mathbb{O} \in \mathfrak{X}/K} T^*_{\mathbb{O}}\mathfrak{X}$
3. $\mathcal{Y} = \bigcup_{\mathbb{O} \in \mathfrak{X}/K} \overline{T^*_{\mathbb{O}}\mathfrak{X}}$ *gives the irred decomp*

the diagonal map in the fiber product: $\varphi^\theta : \mathcal{Y} \to \mathcal{N}_{\mathfrak{k}}$:

$$\varphi^\theta((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^\theta = -y \quad \text{for} \quad ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

we need another map

$$\varphi^{-\theta}((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) = x^{-\theta} = x + y \quad \text{for} \quad ((\mathfrak{p}_1, x), (\mathfrak{q}_1, y)) \in \mathcal{Y}.$$

$\rightsquigarrow$

- φ^θ the generalized Steinberg map and $\operatorname{Im}\varphi^\theta \subset \mathcal{N}_{\mathfrak{k}}$
- $\varphi^{-\theta}$ the exotic Steinberg map and $\operatorname{Im}\varphi^{-\theta} \subset \mathcal{N}_{\mathfrak{s}}$

Generalized/Exotic Steinberg theory IV

$\pi : T^*\mathfrak{X} \to \mathfrak{X}$: bundle map $\rightsquigarrow$ K-equiv double fibration

$$\begin{array}{ccc} & \mathcal{Y} = T^*\mathscr{B}_K \times_{\mathcal{N}^\theta} T^*\mathscr{P}_G & \\ \swarrow^{\pi} & & \searrow^{\varphi^{\pm\theta}} \\ \mathfrak{X} = \mathscr{B}_K \times \mathscr{P}_G & & \mathcal{N}^{\pm\theta} \end{array}$$

Here $\mathcal{N}^\theta = \mathcal{N}_{\mathfrak{k}}$, $\mathcal{N}^{-\theta} = \mathcal{N}_{\mathfrak{s}}$. Using this diagram, we define orbit maps

$$\begin{array}{cccc} \Phi^{\pm\theta} : & \mathfrak{X}/K & \longrightarrow & \mathcal{N}^{\pm\theta}/K \\ & \Cup & & \Cup \\ & \mathbb{O} & \longmapsto & \mathcal{O} \end{array} \qquad \text{by} \qquad \overline{\varphi^{\pm\theta}(\pi^{-1}(\mathbb{O}))} = \overline{\mathcal{O}},$$

where $\mathbb{O}$ is a K-orbit in $\mathfrak{X}$ and $\mathcal{O}$ is a nilpotent K-orbit in $\mathcal{N}^{\pm\theta}$.

$$\text{Since } \pi^{-1}(\mathbb{O}) = T^*_{\mathbb{O}}\mathfrak{X}, \qquad \Phi^{\pm\theta}(\mathbb{O}) = \mathcal{O} \iff \overline{\varphi^{\pm\theta}(T^*_{\mathbb{O}}\mathfrak{X})} = \overline{\mathcal{O}}.$$

Thus $\Phi^{\pm\theta}$ is a map from irred compnts of $\mathcal{Y}$ to nilpotent K orbits (in $\mathfrak{k}$ or $\mathfrak{s}$).

- Φ^θ called the generalized Steinberg map and
- $\Phi^{-\theta}$ called the exotic Steinberg map.

Classical Steinberg map and RS correspondence (review) I

Consider a special but also general case, $G = K$ & $P = Q = B$

Then $\mathfrak{X} = G/B \times G/B = \mathscr{B}_G \times \mathscr{B}_G$, where $\mathscr{B}_G = G/B$ is the full flag variety.

1. $\mathfrak{X}/G \simeq B\backslash G/B \simeq W_G$ via Bruhat decomp
2. Steinberg map: $\Phi : W_G \to \mathcal{N}_{\mathfrak{g}}/G$.
3. $\mathbb{O}_w$: G-orbit through $(B, wB) \in \mathfrak{X}$ $(w \in W_G)$
4. $\mathcal{Y}_w := \overline{\pi^{-1}(\mathbb{O}_w)}$ is an irreducible component of the variety $\mathcal{Y}$ (called Steinberg variety, in this setting).
5. We get $\varphi(\mathcal{Y}_w) = \overline{\mathcal{O}_\lambda} \subset \mathcal{N}_{\mathfrak{g}}$: the closure of a nilpotent orbit $\rightsquigarrow$ $\Phi : W_G \to \mathcal{N}_{\mathfrak{g}}/G$

When $G = \mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$:

$$W_G = \mathfrak{S}_n \quad \& \quad \mathcal{N}_{\mathfrak{g}}/G \simeq \mathscr{P}(n) \text{ (partitions) via Jordan normal form}$$

$\rightsquigarrow$ the Steinberg map:

$$\Phi : \mathfrak{S}_n \ni w \mapsto \lambda \in \mathscr{P}(n).$$

Here the Robinson-Schensted correspondence enters

Classical Steinberg map and RS correspondence (review) II

$$\mathrm{RS} : S_n \xrightarrow{\sim} \coprod_{\lambda \in \mathscr{P}(n)} \{(T_1, T_2) \mid T_i \in \mathrm{STab}_\lambda\},$$

where STab_λ denotes the set of standard tableaux of the shape λ.

Theorem (Steinberg [9])

The Steinberg map

$$\Phi : S_n \ni w \mapsto \lambda \in \mathscr{P}(n)$$

defined by $\varphi(\mathcal{Y}_w) = \overline{\mathcal{O}_\lambda}$ *factors through the Robinson-Schensted correspondence.*

$$\begin{array}{ccccc} S_n & \xrightarrow[\mathrm{RS}]{\sim} & \coprod_{\lambda \in \mathscr{P}(n)} \{(T_1, T_2) \mid T_i \in \mathrm{STab}_\lambda\} & & \ni (T_1, T_2) \\ & \searrow^{\Phi} & \downarrow & & \downarrow \\ & & \mathscr{P}(n) & & \ni \lambda = \mathrm{shape}(T_i) \end{array}$$

Generalized RS correspondence for type AIII I

Recall $\overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r \simeq \mathfrak{X}/K$: pairs of partial permutations classifying K orbits on $\mathfrak{X}$.

This looks like $\mathfrak{S}_n$ in classical case and there exists a generalized RS correspondence to pairs of standard tableaux with decorations.

Notation

Define $\lambda' \Subset \lambda \iff \lambda' \subset \lambda$ & the skew tableau λ/λ' is *column strip*

$\mathscr{P}(n) := \{\lambda \vdash n\}$: the set of partitions of n

Theorem (gen RS correspondence)

$\exists$ *combinatorial bijection between the pairs of partial permutations and pairs of decorated standard tableaux:*

$$\overline{\mathfrak{T}} \xrightarrow{\simeq} \coprod_{(\lambda,\mu)\in\mathscr{P}(p)\times\mathscr{P}(q)} \mathcal{T}_{\lambda,\mu}$$

where $\mathcal{T}_{\lambda,\mu} = \{(T_1, T_2; \lambda', \mu'; \nu)$ satisfying $(*)$ & $(**)$ below$\}$

$(*)$ $(T_1, T_2) \in \mathrm{STab}(\lambda) \times \mathrm{STab}(\mu)$

$(**)$ $\nu \Subset \lambda' \Subset \lambda$, $\nu \Subset \mu' \Subset \mu$ & $|\lambda'| + |\mu'| = |\nu| + r$.

Generalized RS correspondence for type AIII II

As before, we get a commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{X}/K \simeq \mathfrak{T}_{(p,q;r)}/\mathfrak{S}_r & \xrightarrow[\text{genRS}]{\sim} & \displaystyle\coprod_{(\lambda,\mu)\in\mathscr{P}(p)\times\mathscr{P}(q)} \mathcal{T}_{\lambda,\mu} & \ni & (T_1, T_2; \lambda', \mu'; \nu) \\
& \searrow^{\Phi^\theta} & \downarrow & & \downarrow \\
& & \mathscr{P}(p) \times \mathscr{P}(q) & \ni & (\lambda, \mu)
\end{array}
$$

This diagram actually commutes with the Weyl group representations i.e.,

> the fiber of a nilpotent orbit $\mathcal{O}_{(\lambda,\mu)} \subset \mathcal{N}_{\mathfrak{k}}$ inherits a structure of Weyl group representations (Springer correspondence)

However, we don't know a rigorous geometric reason of this phenomenon

Thank you for your attention!!

End of Talk

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Appendix: I

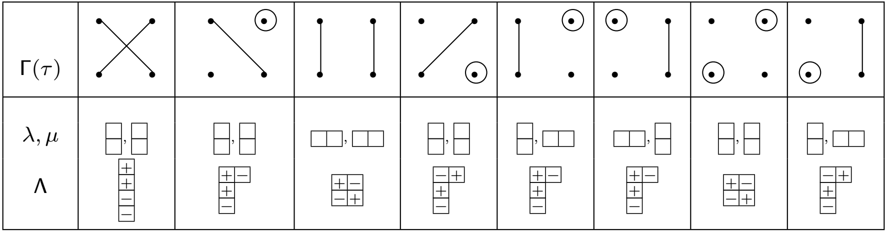
Two Appendices follows:

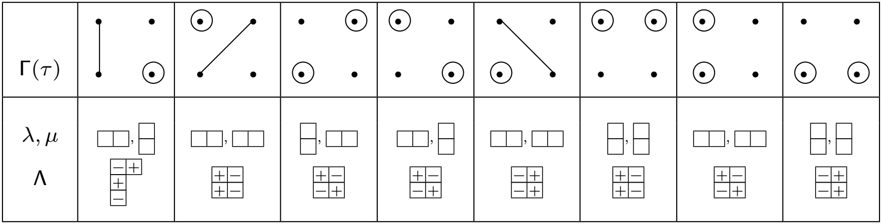
- Table of generalized/exotic Steinberg maps for $p = q = r = 2$
- Recipe of generalized RS correspondence

Table for $p = q = r = 2$: $K = \mathrm{GL}_2 \times \mathrm{GL}_2 \curvearrowright \mathfrak{X} = \mathrm{Gr}_2(\mathbb{C}^4) \times \mathbb{P}^1 \times \mathbb{P}^1$

Table of Steinberg maps for $\tau \in \overline{\mathfrak{T}} \simeq \mathfrak{X}/K$ $(p = q = r = 2)$

- Generalized Steinberg map: $\Phi^{\theta}(\mathbb{O}_\tau) = \mathcal{O}_{(\lambda,\mu)}$ for pair of Young diagrams (λ, μ)
- Exotic Steinberg map: $\Phi^{-\theta}(\mathbb{O}_\tau) = \mathfrak{O}_\Lambda$ for signed Young diagram Λ





Recipe for gen RS correspondence I

Some notations:

- classical RS correspondence: $\mathsf{RS}(\sigma) = (\mathsf{RS}_1(\sigma), \mathsf{RS}_2(\sigma)) \in \mathrm{STab}_\nu^2$
- jeu de taquin of skew tableau: $T * S$ for two standard tableaux, e.g.,

$$\begin{array}{|c|c|}\hline 1&3\\\hline 6\\\cline{1-1}\end{array} * \begin{array}{|c|c|c|}\hline 2&4&5\\\hline 7\\\cline{1-1}\end{array} = \mathrm{Rect}\left(\begin{array}{cccc} & & \boxed{2}&\boxed{4}\,\boxed{5}\\ & & \boxed{7} & \\ \boxed{1}&\boxed{3} & & \\ \boxed{6} & & & \end{array}\right) = \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&7\\\cline{1-2} 6\\\cline{1-1}\end{array}.$$

- $[L], [L'], [M], [M']$ denotes vertical std tableaux whose entries are from L, L', M, M'

Recipe for gen RS correspondence II

Recipe

$\mathrm{genRS}(\tau) = (T_1, T_2; \lambda', \mu'; \nu)$, where

$$(T_1, T_2, \lambda', \mu', \nu) = \Big([L] * \mathrm{RS}_1(\sigma) * [L'], [M] * \mathrm{RS}_2(\sigma) * [M'],$$
$$\mathrm{shape}([L] * \mathrm{RS}_1(\sigma)), \mathrm{shape}([M] * \mathrm{RS}_2(\sigma)), \mathrm{shape}(\mathrm{RS}_1(\sigma))\Big)$$

- $(T_1, T_2) \in \mathrm{STab}_\lambda \times \mathrm{STab}_\mu$
- partitions λ', μ', ν with $\nu \Subset \lambda' \Subset \lambda$, $\nu \Subset \mu' \Subset \mu$, and $|\lambda'| + |\mu'| = |\nu| + r$.

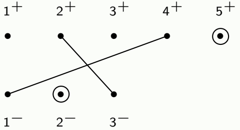
$\mathcal{T}_{\lambda,\mu}$ = (collections of $(T_1, T_2, \lambda', \mu', \nu)$ above)
$\rightsquigarrow$ bijection between the fiber $(\Phi^\theta)^{-1}(\mathcal{O}_{(\lambda,\mu)})$ and $\mathcal{T}_{\lambda,\mu}$.

Recipe for gen RS correspondence III

Example

Recall the example τ before:

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \left(\begin{array}{cccc} \boldsymbol{e}_2 & \boldsymbol{e}_4 & \boldsymbol{e}_5 & 0 \\ \hline \boldsymbol{e}_3 & \boldsymbol{e}_1 & 0 & \boldsymbol{e}_2 \end{array} \right) \rightsquigarrow \Gamma(\tau) =$$



$$I = \{2, 4\}, \quad L = \{5\}, \quad L' = \{1, 3\},$$

$$J = \{1, 3\}, \quad M = \{2\}, \quad M' = \varnothing, \quad \sigma = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in \mathrm{Bij}(J, I).$$

$$\mathrm{genRS}(\tau) = (T_1, T_2, \lambda', \mu', \nu) = \left(\begin{ytableau} 1 & 3 \\ 2 \\ 4 \\ 5 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, \ydiagram{1,1,1}, \ydiagram{2,2}, \ydiagram{1,1} \right).$$