Steinberg variety and moment maps over multiple flag varieties II

—joint work with Hiroyuki Ochiai

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Plan of talk

1. Motivation & Problems
2. Multiple flag variety (= MFV) in classical cases
   Introduce classification of MFV by Magyar-Weymann-Zelevinsky for type A
   Discuss relation to spherical actions, and more results on other types
3. Double flag variety for symmetric pair
   Introduce double flag variety
   Establish criterions for finiteness of orbits
   Discuss representation theoretic meaning of finiteness of orbits
4. Steinberg theory for MFV
   Describe moment maps and nilpotent varieties
5. Example : $U(2,2)$
   Give complete classification of orbits

Problems

G : algebraic group / $\mathbb{C}$ $\quad$ $B \subset G$: Borel subgrp

$G \curvearrowright X = G/B \times G/B$ $\rightsquigarrow$ Steinberg theory (1st Talk)

How to generalize it?

- B : Borel $\rightsquigarrow$ P: parabolic
  well studied including the case of KGP
  (cf Ciubotaru-Trapa-N, arXiv:0903.1039v1 [math.RT])
- Try several copies $X = G/P_1 \times G/P_2 \times \ldots \times G/P_k$
  Almost complete results for classical cases (explained later)
- View point of symmetric pairs
  Only a first step, not so much progress now ...
  But this is our main topic today

Multiple flag variety for type A

$G = \mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$: general linear group

$P \subset G$: parabolic subgrp (up to conjugate) $\longleftrightarrow$ $\lambda \in \mathscr{C}(n)$: composition

(composition = unordered partition)

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathscr{C}(n)$

$$P = P_\lambda = \begin{pmatrix} \boxed{\mathrm{GL}_{\lambda_1}} & & * \\ & \boxed{\mathrm{GL}_{\lambda_2}} & \\ & & \ddots & \\ 0 & & & \boxed{\mathrm{GL}_{\lambda_\ell}} \end{pmatrix}$$

partial flag $\mathscr{F}_\lambda = (F_k)_{0 \le k \le \ell}$ of subspaces

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell \quad \text{s.t.} \quad \dim F_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$\Longrightarrow \; P_\lambda = \mathrm{Stab}_G(\mathscr{F}_\lambda) : \text{ Stabilizer of flag}$$

Notation $\mathfrak{X}_P = G/P$: partial flag variety

Theorem (Magyar-Weymann-Zelevinsky ($G = \mathrm{GL}_n$))

1. *For $P_1, \dots, P_k$: proper parabolics, $\# G \backslash (\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \cdots \times \mathfrak{X}_{P_k}) < \infty$ $\Longrightarrow$ $k \leq 3$.*
2. *$\mathfrak{X}_{P_\lambda} \times \mathfrak{X}_{P_\mu} \times \mathfrak{X}_{P_\nu}$ is of finite type $\Longleftrightarrow$ it is in the table below*

type	$(\ell(\lambda), \ell(\mu), \ell(\nu))$	*extra condition(s)*
$S_{q,r}$	$(2, q, r)$	$\lambda = (n-1, 1)$
D_{r+2}	$(2, 2, r)$	
E_6	$(2, 3, 3)$	
E_7	$(2, 3, 4)$	
E_8	$(2, 3, 5)$	
$E^{(a)}_{r+3}$	$(2, 3, r)$	$\lambda = (n-2, 2)$ $(n \geq 4)$
$E^{(b)}_{r+3}$	$(2, 3, r)$	$\mu = (\mu_1, \mu_2, 1)$

(with possible changes of order of the factors in each λ, μ, ν)

Some remarks in order

$\# G \backslash (\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \cdots \times \mathfrak{X}_{P_k}) < \infty$

- $k = 1$: $\mathfrak{X}_P$ is homogeneous
- $k = 2$: $G \backslash (\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}) \simeq P_1 \backslash G / P_2 \simeq W_{P_1} \backslash W / W_{P_2}$: Bruhat decomposition

  Notation: W_P : Weyl group of P (or its Levi)

Littelmann classified $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ of finite type
when P_1, P_2 : max parabolic & $P_3 = B$: Borel [J. ALg. (1994)]

$$\# G \backslash (\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B) < \infty \iff \# B \backslash (\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}) < \infty$$
$$\iff \mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \text{ is } G\text{-spherical}$$

$P_i \leftrightarrow \varpi_i$ $(i = 1, 2)$: fundamental weight
$\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}$ is G-spherical

$$\iff \quad V_{k\varpi_1} \otimes V_{\ell\varpi_2} \text{ decomposes multiplicity freely as } G\text{-module } (\forall k, \ell \geq 0)$$

Summary of classification

- Magyar-Weymann-Zelevinsky classified MFV of finite type for type A & type C (not intoroduced here)
  They also classified orbits
  [Adv. Math. **141** (1999); J. Algebra **230** (2000)]
- For type B & D, MWZ claims complete classification, but no explicit table available
- For exceptional groups, ∃ result by Popov:
  classification of triple flag varieties with open orbit
  [J. ALg. **313**(2007)]

  Existence of open orbit is necessary for finite type,
  but it does NOT imply finiteness of orbits

Mirabolic (= miraculous parabolic) case

For type A, $\exists$ special wonderful case called mirabolic
$G = \mathrm{GL}_n \supset B$: Borel & $P = P_{(n-1,1)}$: max parabolic (mirabolic)

$$\mathfrak{X}_B \times \mathfrak{X}_B \times \mathfrak{X}_P \simeq \mathscr{F}\ell_n \times \mathscr{F}\ell_n \times \mathbb{P}(\mathbb{C}^n)$$

For this, there are many good properties known due to
Travkin, Finkelberg-Ginzburg-Travkin, Achar-Henderson, Syu Kato, ...

- Analogue of Robinson-Schensted-Knuth algorithm for Springer fiber micro-local cells and action of Hecke algebra, etc.
- Enhanced nilpotent cone and orbits on $\mathcal{N}(\mathfrak{g}) \times \mathbb{C}^n$,
  local intersection theory (IC complexes) on the closure of nilpotent orbits
- Exotic nilpotent cone and orbits,
  Springer representations for BC-type Weyl group, Kazhudan-Lusztig theory, ...

Double flag variety — definition

G : reductive alg grp $/\mathbb{C}$
$\theta \in \mathrm{Aut}\, G$: involution
$\rightsquigarrow$ $K = G^\theta$: symmetric subgrp ($\fallingdotseq$ $\mathbb{C}$-fication of max cpt subgrp)

P : parabolic & P' : θ-stable parabolic of G
$\rightsquigarrow$ $Q := P' \cap K$: parabolic of K

Remark

For $\forall Q \subset K$: parabolic, $\exists\ P' \subset G$: θ-stable parabolic s.t. $Q = P' \cap K$

Notation

$\mathfrak{X}_P := G/P$: partial flag var & $\mathfrak{X}_P^\theta := \mathfrak{X}_{\theta(P)} = G/\theta(P)$
$\mathcal{Z}_Q := K/Q$: partial flag for K

$\mathfrak{X}_P \times \mathcal{Z}_Q$: double flag variety $\curvearrowleft K$ acts diagonally

Relation to MFV for G

Remark

1. Triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ with diag G-action is a special case of double flag variety $\mathfrak{X}_P \times \mathcal{Z}_Q$ with K-action

   ($\because$ Take $\mathbb{G} = G \times G$ and $\mathbb{K} = \Delta G$ as usual)

2. $\mathcal{Z}_Q \simeq K \cdot P' / P' \xhookrightarrow{\text{closed}} \mathfrak{X}_{P'}$ i.e. $\mathcal{Z}_Q$ is a closed K-orbit in $K \backslash \mathfrak{X}_{P'}$

   Thus we get a closed embedding:

   $$\mathfrak{X}_P \times \mathcal{Z}_Q \xhookrightarrow{\text{closed}} \mathfrak{X}_P \times \mathfrak{X}_{P'} \text{ with diag } K\text{-action}$$

   In general $\#K \backslash (\mathfrak{X}_P \times \mathfrak{X}_{P'}) = \infty$ however

Finiteness of orbits

Theorem (N-Ochiai)

$\# G \backslash (\mathfrak{X}_P \times \mathfrak{X}_P^\theta \times \mathfrak{X}_{P'}) < \infty \implies \# K \backslash (\mathfrak{X}_P \times \mathcal{Z}_Q) < \infty$

Corollary

P : parabolic in G

$\mathfrak{X}_P \times \mathfrak{X}_P^\theta$: G-spherical variety $\implies \mathfrak{X}_P$: K-spherical variety

Proof of Corollary.

$\exists\, B$: θ-stable Borel s.t. $S := K \cap B$ is Borel for K

$\mathfrak{X}_P \times \mathfrak{X}_P^\theta$: G-spherical variety

$\iff \# B \backslash (\mathfrak{X}_P \times \mathfrak{X}_P^\theta) < \infty \iff \# G \backslash (\mathfrak{X}_P \times \mathfrak{X}_P^\theta \times \mathfrak{X}_B) < \infty$

$\xRightarrow{\text{Theorem}} \# K \backslash (G/P \times K/S) < \infty \iff \# S \backslash G/P < \infty$ □

$\iff \mathfrak{X}_P = G/P$ is K-spherical

Representation theoretic meaning of Corollary (1)

Δ^+ : positive roots $\supset \Pi$: simple roots $\supset \Phi$: subset (parabolic data)
Define $\lambda := \sum_{\alpha\in\Phi} \omega_\alpha$ (ω_α : fund weight $\leftrightarrow \alpha$)

V_λ : finite dim irred rep $\ni v_\lambda$: highest weight vector

$$P = \{g \in G \mid g \cdot v_\lambda \in \mathbb{C} v_\lambda\} : \text{parabolic} \longleftrightarrow \Phi \subset \Pi$$

$$[v_\lambda] \in \mathbb{P}(V_\lambda) : \text{proj space} \rightsquigarrow \mathfrak{X}_P \simeq G \cdot [v_\lambda]$$

$\widehat{\mathfrak{X}}_P := \overline{G v_\lambda} \subset V_\lambda$: affine cone $/\mathfrak{X}_P$ called highest weight variety

$$\rightsquigarrow \mathbb{C}[\widehat{\mathfrak{X}}_P] \simeq \bigoplus_{\ell\geq 0} V_{\ell\lambda} \ \&$$

$$\mathbb{C}[\widehat{\mathfrak{X}}_P \times \widehat{\mathfrak{X}}_{\theta(P)}] \simeq \bigoplus_{k,\ell\geq 0} V_{k\lambda} \otimes V_{\ell\lambda^\theta}$$

: multiplicity free (= MF) decomposition

Representation theoretic meaning of Corollary (2)

Lemma

1. $\mathfrak{X}_P$ *is K-spherical* $\iff$ $V_{\ell\lambda}|_K$ $(\forall \ell \geq 0)$ *is a MF K-module*
2. $\mathfrak{X}_P \times \mathfrak{X}_P^\theta$ *is G-spherical*
$$\iff V_{k\lambda} \otimes V_{\ell\lambda^\theta} \ (\forall k, \ell \geq 0) \text{ is a MF } G\text{-module}$$

Proof of (1).

$$\mathfrak{X}_P \text{ is } K\text{-spherical} \iff \widehat{\mathfrak{X}}_P \text{ is } \mathbb{C}^\times \times K\text{-spherical}$$
$$\iff \mathbb{C}[\widehat{\mathfrak{X}}_P] \text{ is a MF } (\mathbb{C}^\times \times K)\text{-module}$$
$$\iff V_{\ell\lambda}|_K \ (\forall \ell \geq 0) \text{ is a MF } K\text{-module}$$

□

Corollary

$V_{k\lambda} \otimes V_{\ell\lambda^\theta}$ $(\forall k, \ell \geq 0)$ *decomposes MF as a G-module*

$\implies$ $V_{m\lambda}|_K$ $(\forall m \geq 0)$ *decomposes MF as a K-module*

How to prove Theorem?

Theorem (mentioned above, quoted again)

$\# G\backslash(\mathfrak{X}_P \times \mathfrak{X}_P^\theta \times \mathfrak{X}_{P'}) < \infty \implies \# K\backslash(\mathfrak{X}_P \times \mathcal{Z}_Q) < \infty$

$P' = G \implies Q = K$ and theorem reduces to the well-known
$\# K\backslash G/P < \infty$ (Wolf, Matsuki, Rossmann, Springer, ...)

$\exists$ beautiful proof by Miličić in his lecture note, available online
$\rightsquigarrow$ Apply his idea to $K\backslash(\mathfrak{X}_P \times \mathcal{Z}_Q)$

Key idea: θ-twisted diagonal embedding:

$$\Delta_\theta : \mathfrak{X}_P \ni P_1 \mapsto (P_1, \theta(P_1)) \in \mathfrak{X}_P \times \mathfrak{X}_P^\theta$$

$$\mathfrak{X}_P \times \mathcal{Z}_Q \xrightarrow{\sim} \Delta_\theta(\mathfrak{X}_P) \times \mathcal{Z}_Q \hookrightarrow \Delta_\theta(\mathfrak{X}_P) \times \mathfrak{X}_{P'} \subset \mathfrak{X}_P \times \mathfrak{X}_P^\theta \times \mathfrak{X}_{P'}$$

Δ_θ-twisted action gives Bruhat decomposition:

$$\Delta_\theta(G)\backslash\big(\Delta_\theta(\mathfrak{X}_P) \times \mathfrak{X}_{P'}\big) \simeq G\backslash\big(\mathfrak{X}_P \times \mathfrak{X}_{P'}\big) \simeq W_P\backslash W/W_{P'}$$

Proof of Theorem.

Pick $\Delta_\theta(G)$-orbit $\mathcal{O}^\theta_w \in \Delta_\theta(G)\backslash(\Delta_\theta(\mathfrak{X}_P)\times\mathfrak{X}_{P'})$ $\quad (w \in W_P\backslash W/W_{P'})$

Lemma (Key Lemma)

$\forall\mathcal{O} \in G\backslash(\mathfrak{X}_P\times\mathfrak{X}^\theta_P\times\mathfrak{X}_{P'})$, $X := \Delta_\theta(\mathfrak{X}_P)\times\mathcal{Z}_Q$

1. $\#K\backslash(\mathcal{O}\cap\mathcal{O}^\theta_w\cap X) < \infty$
2. $\mathcal{O}\cap\mathcal{O}^\theta_w\cap X = \sqcup_{i=1}^{\ell}\mathbb{O}_i$: *K-orbit decomposition*
   $\Longrightarrow$ $\mathbb{O}_i$ *is a connected component of* $\mathcal{O}\cap\mathcal{O}^\theta_w\cap X$

Let us assume the above lemma. Since

- decomposition $X = \sqcup_{w\in W_P\backslash W/W_{P'}}\mathcal{O}^\theta_w\cap X$ is finite
- finitely many G-orbits $\mathcal{O}$ in $\mathfrak{X}_P\times\mathfrak{X}^\theta_P\times\mathfrak{X}_{P'}$ by the assumption

we conclude that $\#K\backslash X = \#K\backslash(\mathfrak{X}_P\times\mathcal{Z}_Q) < \infty$.

□

Theorem does not exhaust double flag variety of finite type
Introduce another technique, which can produce more examples.

Key idea

Embed G/Q into $G/P_2 \times G/P_3$: product of (partial) flag varieties

A generalization of Harish-Chandra embedding:

$$G/K \hookrightarrow \text{(product of flag varieties)}$$

Example (Classical Harish-Chandra embedding)

Assume $K = P \cap P^\circ$ for P parabolic and its opposite P°

$$\rightsquigarrow \; G/K \ni gK \mapsto (gP, gP^\circ) \in \mathfrak{X}_P \times \mathfrak{X}_{P^\circ} : \text{open embedding}$$

Thus we get:

$$B \backslash G/K \hookrightarrow B \backslash (\mathfrak{X}_P \times \mathfrak{X}_{P^\circ}) \simeq G \backslash (\mathfrak{X}_B \times \mathfrak{X}_P \times \mathfrak{X}_{P^\circ})$$

$$\# B \backslash G/K < \infty \iff \exists \text{ open } B\text{-orbit} \iff \# B \backslash (\mathfrak{X}_P \times \mathfrak{X}_{P^\circ}) < \infty$$
$$\iff \# G \backslash (\mathfrak{X}_B \times \mathfrak{X}_P \times \mathfrak{X}_{P^\circ}) < \infty$$

Suggests simpler & easier criterion of $\#K\backslash(\mathfrak{X}_P \times \mathcal{Z}_Q) < \infty$

Proposition

P_i $(i = 1, 2, 3)$: parabolic subgrp of G satisfying

1. *$Q := P_2 \cap P_3$ is a parabolic of K*
2. $\#G\backslash(\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}) < \infty$

$\Longrightarrow$ $\#K\backslash(\mathfrak{X}_{P_1} \times \mathcal{Z}_Q) < \infty$

Proof.

By (1), $\exists$ diag embedding $G/Q \hookrightarrow \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$; $gQ \mapsto (gP_2, gP_3)$
From this:

$$
\begin{aligned}
K\backslash(\mathfrak{X}_{P_1} \times \mathcal{Z}_Q) &\cong P_1\backslash G/Q \\
&= G\backslash(\mathfrak{X}_{P_1} \times G/Q) \hookrightarrow G\backslash(\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3})
\end{aligned}
$$

By (2), $\#G\backslash(\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}) < \infty$ □

MFV of finite type (type A) — Tables

Type AI : $G/K = \mathrm{SL}_n/\mathrm{SO}_n$ $(n \geq 3)$

P	Q	$\mathfrak{X}_P$	$\mathcal{Z}_Q$	extra condition
maximal	any	$\mathrm{Grass}_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$	
$(\lambda_1, \lambda_2, \lambda_3)$	Siegel	$\mathfrak{X}_P$	$\mathrm{LGrass}(\mathbb{C}^n)$	n is even

Type AII : $G/K = \mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ $(n \geq 2)$

P	Q	$\mathfrak{X}_P$	$\mathcal{Z}_Q$
maximal	any	$\mathrm{Grass}_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$
$(\lambda_1, \lambda_2, \lambda_3)$	Siegel	$\mathfrak{X}_P$	$\mathrm{LGrass}_m(\mathbb{C}^{2n})$

Type AIII : $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$ $(n = p + q)$

P	Q_1	Q_2	$\mathfrak{X}_P$	$\mathcal{Z}_Q$
any	mirabolic	GL_q	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^p)$
any	GL_p	mirabolic	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^q)$
maximal	any	any	$\mathrm{Grass}_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$
$(\lambda_1, \lambda_2, \lambda_3)$	maximal	maximal	$\mathfrak{X}_P$	$\mathrm{Grass}_k(\mathbb{C}^p) \times \mathrm{Grass}_\ell(\mathbb{C}^q)$

Moment maps

$X := \mathfrak{X}_P \times \mathcal{Z}_Q \curvearrowleft K$: diag K-action

Want to apply Steinberg theory to $K \backslash X$:

$$
\begin{array}{ccccc}
T^*X & = & T^*\mathfrak{X}_P \times T^*\mathcal{Z}_Q & \ni & ((\mathfrak{p}', \xi), (\mathfrak{q}', \eta)) \\
 & & \downarrow \mu_{\mathfrak{X}_P} \times \mu_{\mathcal{Z}_Q} & & \downarrow \\
 & \searrow \mu_X & \mathfrak{g}^* \times \mathfrak{k}^* & \ni & (\xi, \eta) \\
 & & \downarrow \alpha & & \downarrow \\
 & & \mathfrak{k}^* & \ni & \xi|_{\mathfrak{k}} + \eta
\end{array}
$$

$\mu_{\mathfrak{X}_P}(T^*\mathfrak{X}_P) = G \cdot \mathfrak{u}_P = \overline{\mathscr{O}_P^G} \subset \mathcal{N}(\mathfrak{g})$: Richardson orbit for P

$\mu_{\mathcal{Z}_Q}(T^*\mathcal{Z}_Q) = K \cdot \mathfrak{u}_Q = \overline{\mathscr{O}_Q^K} \subset \mathcal{N}(\mathfrak{k})$: Richardson orbit for Q

Steinberg variety for MFV $X = \mathfrak{X}_P \times \mathcal{Z}_Q$

$S_X := \mu_X^{-1}(0) = \bigcup_{\mathbb{O} \in K \backslash} \overline{T^*_{\mathbb{O}} X}$: Steinberg variety

Notation: $x^\theta := \frac{1}{2}(x + \theta(x)) \in \mathfrak{k}$

$$\begin{array}{ccc} \mathfrak{g} \ni x & \longleftrightarrow & \xi \in \mathfrak{g}^* \\ x^\theta & \longleftrightarrow & \xi|_{\mathfrak{k}} \end{array}$$

$$\begin{aligned}(\mu_{G/P} \times \mu_{K/Q})(S_X) &= \{(x, y) \in \mathfrak{g} \times \mathfrak{k} \mid x \in \overline{\mathscr{O}_P^G},\ y \in \overline{\mathscr{O}_Q^K},\ \tfrac{1}{2}(x + \theta(x)) + y = 0\} \\ &= \{(x, -x^\theta) \in \mathfrak{g} \times \mathfrak{k} \mid x \in \overline{\mathscr{O}_P^G},\ x^\theta \in \overline{\mathscr{O}_Q^K}\} \\ &\simeq \{x \in \mathfrak{g} \mid x \in \overline{\mathscr{O}_P^G},\ x^\theta \in \overline{\mathscr{O}_Q^K}\}\end{aligned}$$

Definition

$$\mathcal{N}_{\mathfrak{X}_P \times \mathcal{Z}_Q} := \{x \in \mathfrak{g} \mid x \in \overline{\mathscr{O}_P^G}, x^\theta \in \overline{\mathscr{O}_Q^K}\}$$

: nilpotent variety for double flag variety

$$\mathcal{N}_{\mathfrak{X}_P \times \mathcal{Z}_Q} = \{x \in \mathfrak{g} \mid x \in \overline{\mathscr{O}_P^G}, x^\theta \in \overline{\mathscr{O}_Q^K}\} : \text{nilpotent variety}$$

Naïve questions

1. $\# K \backslash \mathcal{N}_{\mathfrak{X}_P \times \mathcal{Z}_Q} < \infty$? [Answer : NO]
2. Geometric structure of K-stable, irreducible closed subvariety $(\mu_{G/P} \times \mu_{K/Q})(\overline{T^*_{\mathbb{O}} X}) =: \mathcal{N}_{\mathfrak{X}_P \times \mathcal{Z}_Q}(\mathbb{O})$?
3. Geometric cells on $K \backslash \mathfrak{X}_P \times \mathcal{Z}_Q$? Closure relations etc.
4. How it can be related to representation theory?

We give an easiest example in type A

$U(2,2)$

$G = \mathrm{GL}_4(\mathbb{C}) \supset K = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$
$V = \mathbb{C}^4 \qquad V = V^+ \oplus V^- \qquad V^\pm = \mathbb{C}^2$

$P = P_{(2,2)}$: max parabolic in G
$Q = Q^+ \times Q^-$: product of Borel subgrps of $\mathrm{GL}(V^\pm)$

$$\begin{aligned} \rightsquigarrow \quad \mathfrak{X}_P \times \mathcal{Z}_Q &= (\mathrm{GL}_n/P)\times(\mathrm{GL}_2/B)\times(\mathrm{GL}_2/B) \\ &\simeq \mathrm{Grass}_2(\mathbb{C}^4)\times\mathbb{P}(\mathbb{C}^2)\times\mathbb{P}(\mathbb{C}^2) \ni (L, p_1, p_2) \end{aligned}$$

Thus we have

1. In the whole projective space $\mathbb{P}(\mathbb{C}^4) = \mathbb{P}(V)$ of dim $= 3$
2. Two separate lines $[V^\pm]$ which determines the symmetric pair G/K
3. One line $L \in \mathrm{Grass}_2(\mathbb{C}^4)$
4. Two pints p_1, p_2 in $\mathrm{Grass}_2(V^+)$ and $\mathrm{Grass}_2(V^-)$ respectively

$$K\backslash\mathfrak{X}_P\times\mathcal{Z}_Q \quad \longleftrightarrow \quad \text{configurations of } (L, p_1, p_2) \text{ inside } \mathbb{P}(V)$$

$V = \mathbb{C}^4 \qquad V = V^+ \oplus V^- \qquad \dim V^\pm = 2$

$(L \subset V,\ p_1 \subset V^+,\ p_2 \subset V^-)$ s.t. $\dim L = 2,\ \dim p_1 = \dim p_2 = 1$

Lemma

1. *Configurations* (L, p_1, p_2) *inside* V *are classified by dimensions*

$$(\ell^+, \ell^-) = (\dim L \cap V^+, \dim L \cap V^-)$$

$$[\ell; w^+ w^-] = [\dim L \cap (p_1 \oplus p_2); \dim L \cap p_1, \dim L \cap p_2]$$

up to $K = \mathrm{GL}_2 \times \mathrm{GL}_2$ *conjugacy*

2. $\# K \backslash \mathfrak{X}_P \times \mathcal{Z}_Q = 14$

Remark

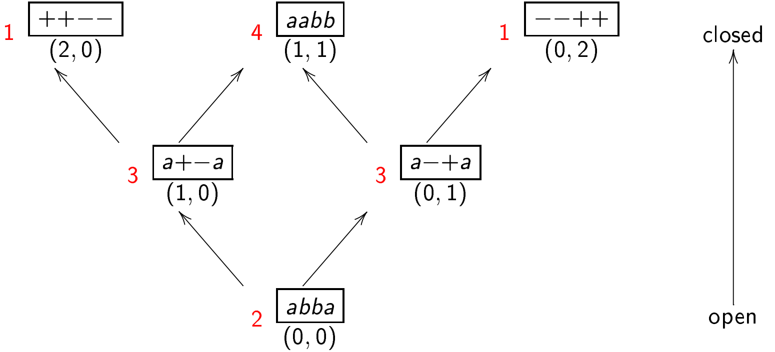
Lemma is misleading but instructive in the sense:

- Need more dimensions for $U(p,q)$
- More dimensions seem to be enough for $U(p,q)$ and general Q

Projection to $K\backslash G/P$

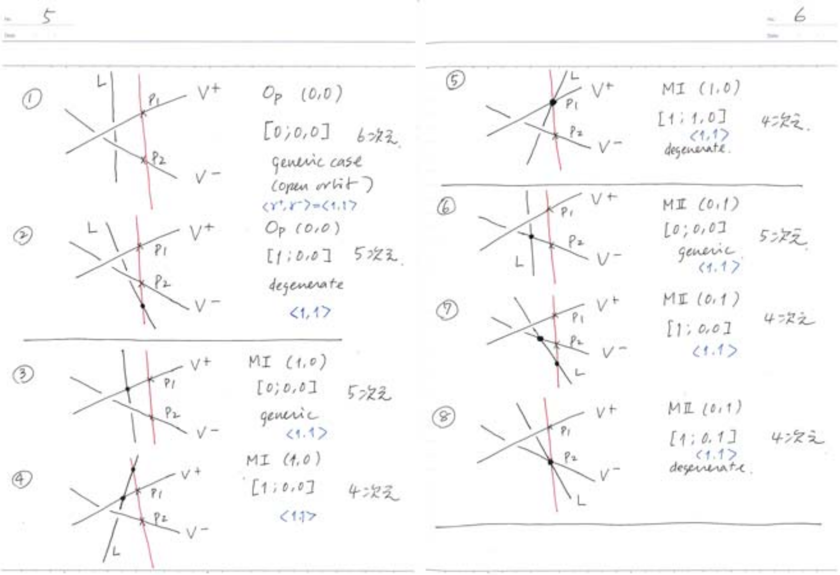
proj : $\mathfrak{X}_P \times \mathcal{Z}_Q \to \mathfrak{X}_P$: K-equivariant (forget Q)

Figure : $K\backslash G/P$ parametrized by $(\ell^+, \ell^-) = (\dim L \cap V^+, \dim L \cap V^-)$

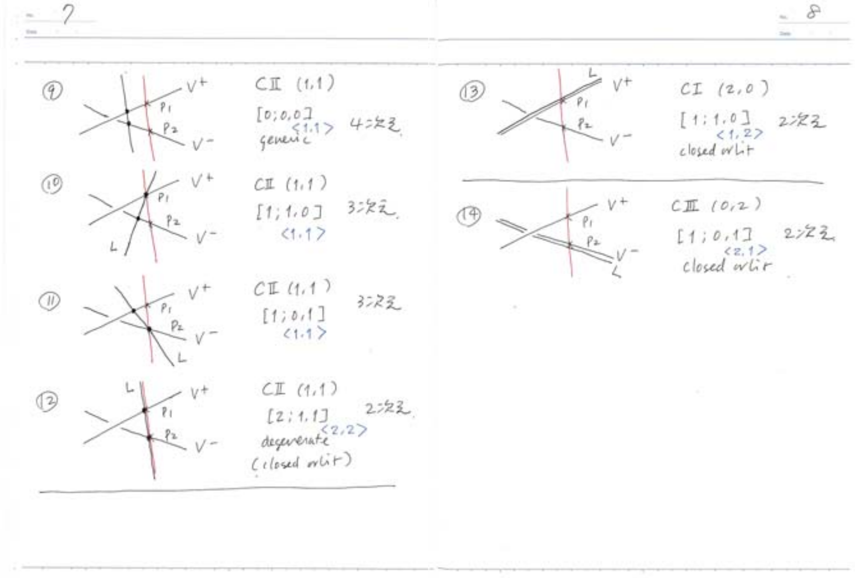


n : # of fibers of proj : $K\backslash\mathfrak{X}_P \times \mathcal{Z}_Q \to K\backslash\mathfrak{X}_P$

$\mathrm{Grass}_2(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2)$ (I)



$\mathrm{Grass}_2(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2)$ (II)



Thank you!!